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On the geometry of reduced cotangent bundles at zero momentum

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Abstract

We consider the problem of cotangent bundle reduction for proper non-free group actions at zero momentum. We show that in this context the symplectic stratification obtained by Sjamaar and Lerman refines in two ways: (i) each symplectic stratum admits a stratification which we call the secondary stratification with two distinct types of pieces, one of which is open and dense and symplectomorphic to a cotangent bundle; (ii) the reduced space at zero momentum admits a finer stratification than the symplectic one into pieces that are coisotropic in their respective symplectic strata.
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1. Introduction

This paper addresses the problem of symplectic reduction for cotangent bundles with proper actions, at zero momentum. From the point of view of mechanics, cotangent bundles are the most important symplectic manifolds since they are the phase spaces for most classical mechanical systems. The geometry of the reduced space plays a crucial role in understanding the dynamics of reduced Hamiltonian systems with non-freely acting symmetry groups. We view this problem, then, as a fundamental one in the theory of geometric mechanics and symplectic reduction.

A general theory of symplectic reduction for proper and non-free actions has been a subject of active research since the original theory was worked out in Marsden and Weinstein [15] and Meyer [16]. The geometric structure of the reduced spaces was first satisfactorily understood, for the case of compact symmetry groups, in the breakthrough paper of Sjamaar and Lerman [24], where the tools of stratification by orbit types were first introduced to precisely determine how the reduced space, which is not in general a manifold, is decomposed into symplectic manifolds called symplectic strata. Indeed, from this point of view, they were able to put in geometric context the earlier work on this problem by proving that the symplectic strata of the reduced space are the symplectic leaves of the reduced Poisson algebra as determined in Arms et al. [2]. These symplectic strata are obtained by first intersecting the zero level set of the momentum map with the points in the original symplectic manifold with the same orbit type, and then taking the

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quotient of this space by the group action. They also explain how the strata fit together by examining the behavior of a linear symplectic action on a symplectic normal space, and applying the Symplectic Slice Theorem due to Marle, Guillemin, and Sternberg.

Since this work, the field has continued to develop substantially. In Bates and Lerman [3], the theory was extended to proper group actions and nonzero momentum, by way of orbit reduction, with the assumption of locally closed coadjoint orbits. In Ortega and Ratiu [19], the theory of Poisson reduction by a free Poisson action, given in Marsden and Ratiu [14], is extended to the singular case. The symplectic reduction theory is extended to the case of non-locally closed coadjoint orbits in Cushman and Śniatycki [6] by looking at accessible sets of invariant Hamiltonian vector fields. A comprehensive reference for all these results, including several generalizations and improvements of the theory and also their consequences in terms of reduction and reconstruction of Hamiltonian dynamics, is found in Ortega and Ratiu [18]. Another text, Cushman and Bates [5], besides giving an overview of the general theory, also contains many computed examples using invariant theory.

Specializing to cotangent bundles, one expects, as in the free case, that the reduced space will admit special structure. Indeed, in the free case, as is well known, the reduced space at zero momentum is in fact simply the cotangent bundle of the orbit space of the base with its canonical symplectic form. At nonzero momentum it is known that the reduced symplectic space is symplectomorphic to a coadjoint orbit bundle (see Marsden and Perlmutter [13]). Alternatively it can be seen as the image of a symplectic embedding into an appropriate cotangent bundle (see for instance Marsden [12]).

Although various attempts were made to apply the general theory of singular reduction to understand the important case of cotangent bundles, until now, there has not been a complete picture without strong assumptions. The literature begins with a result due to Montgomery [17] prior to the work of Sjamaar and Lerman in which he extends the embedding theory of regular cotangent bundle reduction to the case where the involved groups satisfy a special dimension condition and the proper action on the base manifold is assumed to consist of only one orbit type. In the paper [8], Emmrich and Römer give a complete solution to the zero momentum reduced space for a proper action again with the assumption that the base action consists of just one orbit type. As one might guess from the free theory of cotangent bundle reduction and the fact that the orbit space for the base action is a manifold, they obtain that the reduced space at zero momentum is just the cotangent bundle of the orbit space with its canonical symplectic form.

The next paper to address the problem of reduction of cotangent bundles is Lerman et al. [10], where the example of S^1 acting on T^*S^2 is computed and the reduced space at zero momentum is shown to be the "canoe". They also provide a result for singular cotangent bundle reduction at zero in the case that the action admits a cross section.

Finally, we note that in Schmah [23], the results obtained in [8] are again obtained with a different proof and extended, under the same hypothesis on the isotropy groups in the base, to deal with reduction at momentum values with trivial coadjoint orbits.

The main results

There are several new results in this paper. An important guiding principle in this work is that zero momentum reduced data should correspond with data constructed from the group action on the base, in particular the isotropy lattice.

We will consider a proper action of a Lie group G on a manifold M and its lifted action on T^*M, which is Hamiltonian with respect to the canonical symplectic form with equivariant momentum map $\mathbf{J} : T^*M \rightarrow \mathfrak{g}^*$. Our first main result, Theorem 5, is that the isotropy lattice for the G-action on the zero momentum level set, $\mathbf{J}^{-1}(0)$, is isomorphic to the isotropy lattice for the base action of G on M. We obtain this, roughly, by decomposing $\mathbf{J}^{-1}(0)$ as a disjoint union of fiber bundles along the base orbit types and then using a subtle application of the Tube Theorem for slices. Next, relative to this primary decomposition of $\mathbf{J}^{-1}(0)$, knowing its isotropy lattice, we consider for each isotropy type (L) the set $(\mathbf{J}^{-1}(0))_{(L)}$. Call a pair of elements, (H), (L), in the isotropy lattice of M a *connectable pair* over (L), provided $(H) \geq (L)$. This means that L is conjugate to a subgroup of H. Let us denote this relationship by $H \rightarrow L$. We are now able to obtain a decomposition of the manifold, $(\mathbf{J}^{-1}(0))_{(L)}$, into fiber bundles, one for each connectable pair over (L). That is, to each group larger than or equal to (L) in the base lattice we construct a fiber bundle, contained in $(\mathbf{J}^{-1}(0))_{(L)}$.

The symplectic strata of the reduced space $\mathcal{P}_0 := \mathbf{J}^{-1}(0)/G$ are given by $(\mathbf{J}^{-1}(0))_{(L)}/G$ for each (L) in the base isotropy lattice and we will further demonstrate that each of these is in turn stratified by fiber bundles, which we

call seams, one for each connectable pair $H \to L$ over (L). The pair $L \to L$ is in fact identified under a natural diffeomorphism to the cotangent bundle $T^*(M_{(L)}/G)$ and we will prove that this is an open dense piece in this *secondary* stratification of each symplectic stratum. The other pieces fiber over the strata in the boundary of $M_{(L)}/G$.

The reduced symplectic structure fits together with respect to this stratification in an elegant way. The cotangent bundle within each symplectic stratum is open and dense. We prove in Proposition 6 that the restriction of the reduced symplectic form to each seam is in fact equal to the pull back of the corresponding canonical symplectic form of the corresponding cotangent bundle. In Theorem 8 we characterize the reduced symplectic form on each symplectic stratum as the unique extension of the canonical symplectic form of the open and dense cotangent bundle (corresponding to the $L \to L$ connectable pair) to its closure. Furthermore, we prove in Theorem 8 that the seams (corresponding to the $H \to L$ pairs) are in fact coisotropic submanifolds within their corresponding symplectic strata.

We consider the topology of the total reduced space $\mathcal{P}_0$ and obtain a *coisotropic* stratification (Theorem 10) which demonstrates that the full collection of objects, seams and cotangent bundles, corresponding to the entire set of connectable pairs in the isotropy lattice of the base, forms a stratification of $\mathcal{P}_0$, which is of second order in the sense that each of its strata is labelled by a connectable pair in the isotropy lattice. It is finer than the stratification induced by the symplectic strata of Sjamaar and Lerman and, in opposition to the latter, the continuous surjective projection to M/G happens to be a morphism of stratified spaces with respect to the coisotropic stratification of $\mathcal{P}_0$ and the orbit type stratification of M/G.

It should be noted that although the secondary and coisotropic stratifications introduced here are suitable for explaining the bundle structure of the reduced space for cotangent-lifted actions, they lose some of the good properties enjoyed by the symplectic stratification. First, the secondary and coisotropic stratifications are not known to have the conical property, or to satisfy the Whitney conditions as the symplectic stratification (see [24,18]). For general symplectic manifolds, this is possible due to the existence of the Symplectic Slice Theorem, which is not well adapted to the cotangent bundle case. Second, unlike for the symplectic stratification, the secondary pieces of the stratifications introduced here are not invariant under the reduced Hamiltonian flows, which makes our results difficult to apply to dynamics. However, these results do give qualitative information about the evolution of isotropy in the projection of the reduced flows to the reduced base space M/G. See the end of the paper for more remarks on these two comments.

For most of the derivations of our results about these stratifications we will work in the slightly weaker category of Σ-decomposed spaces, because it is computationally simpler. This category is introduced in Section 2. In Section 5, however, we show how these results persist in the category of stratified spaces.

2. Background and preliminaries

The main aim of this section is to review the results on proper group actions and symplectic reduction that we shall need for the rest of the paper. This review will also serve to fix notation. We first review the basic results on proper group actions on manifolds, namely the decomposition of the manifold into orbit types which is a Σ-decomposition (to be introduced later) of the manifold. We then recall the general theory of symplectic reduction at zero momentum for proper group actions which describes the decomposition of the reduced space at zero into symplectic Σ-manifolds obtained in a natural way from the orbit type decomposition of $\mathbf{J}^{-1}(0)$ (see [24]). Finally, we will summarize the known results for cotangent bundle reduction, first in the free case, and then, the next easiest case for proper actions: the case with only one orbit type on the base manifold.

2.1. Σ-Decompositions and proper actions

Recall that a smooth action of a Lie group on a manifold M is *proper* if the map $G \times M \to M \times M$, $(g, m) \mapsto (m, g \cdot m)$ is proper (the inverse image of a compact set is compact). Notice that we have denoted the action map $G \times M \to M$ by a dot. For the proofs of the following key properties see for instance Duistermaat and Kolk [7] or Pflaum [21].

Properties of proper actions

Let M be a G-manifold with a proper action. Then,

1. The isotropy subgroup G_m of any point $m \in M$ is compact.
2. Each orbit $G \cdot m, m \in M$, is a closed submanifold of M diffeomorphic to G/G_m.

3. The orbit space M/G is Hausdorff, locally compact and paracompact.
4. M admits a G-invariant Riemannian metric.
5. If all the isotropy groups of points in M are conjugate to a given one, the orbit space M/G is a smooth manifold and the projection $M \to M/G$ is a surjective submersion.

An important result for proper actions is the standard model for G-invariant neighborhoods. This is a consequence of the existence of slices due to Koszul [9] in the case of G compact and later extended to proper actions by Palais [20]. Let exp be the exponential map associated to a G-invariant metric and S_m the orthogonal complement to $\mathfrak{g} \cdot m = T_m(G \cdot m)$. Consider the product $G \times S_m$ with the left diagonal action of G_m given by $h \cdot (g, v) := (gh^{-1}, h \cdot v)$. This is well defined because by construction S_m is G_m-invariant. This action is free since it is free in the first factor. Next, construct the associated bundle $G \times_{G_m} S_m$ to the principal bundle $G \to G/G_m$. There is a well defined G-action on this bundle given by

$$g \cdot [h, u] = [gh, u].$$

With these constructions, one then has the following result providing an explicit realization of a G-invariant tubular neighborhood of the orbit through m.

Theorem 1 (*Tube Theorem*). *The map* $\phi : G \times_{G_m} S_m \to M$ *given by*

$$\phi([g, u]) = g \cdot \exp_m(u)$$

restricts to a G-equivariant diffeomorphism from a G-invariant neighborhood of the zero section of $G \times_{G_m} S_m$ to a G-invariant neighborhood of $G \cdot m$ in M satisfying

$$\phi([e, 0]) = m.$$

Consequently ϕ maps the set $[G, 0]$, the zero section of the bundle $G \times_{G_m} S_m$, to the orbit $G \cdot m$.

Remark 1. We can construct the G-invariant neighborhood of the zero section of the associated bundle of the previous theorem as follows. Let r be some positive radius smaller than the injectivity radius of $\exp_m$. Then the ball B_r around 0 in S_m is G_m-invariant since the action is by isometries. We refer to B_r and $\exp_m(B_r)$ as a linear slice and a slice through m for the G-action respectively. It is easy to see that the $\exp_m$ map restricted to B_r is a G_m-equivariant diffeomorphism with respect to the linear action of G_m on B_r and the base action of G_m on $\exp_m(B_r)$ since the G_m action must take geodesics to geodesics. Notice, then, that the only group elements leaving the slice invariant are those in G_m, i.e., we have

$$\text{For any } z \in \exp_m(B_r), \quad G_z \subseteq G_m. \tag{1}$$

The G-invariant neighborhood of the zero section, alluded to in the previous theorem, is then $G \times_{G_m} B_r$. The details of the proof of the existence of slices for proper actions and of the Tube Theorem can be found in [7]. ▲

For a subgroup H of a Lie group G the conjugacy class of H consists of all subgroups of G that are conjugate to H and will be denoted by (H). Denote by I_M the set of conjugacy classes of isotropy groups of points of M. Corresponding to each element of this set $(H) \in I_M$ we have the subset of M of orbit type (H) defined by

$$M_{(H)} := \{m \in M : G_m \in (H)\}.$$

For a proper G action on a manifold M the connected components of the orbit type $M_{(H)}$ are embedded submanifolds.

In the set of conjugacy classes of G we can define a partial ordering $\leq$ by $(H) \leq (K)$ if and only if H is conjugate to a subgroup of K in G. We will use the notation $(H) < (K)$ to mean that H is conjugate to a proper subgroup of K in G, i.e., strictly smaller than K. We will represent I_M as a lattice in the following way: we draw an arrow from H to K when H and K are representatives of two classes in I_M such that $(H) < (K)$ and there is no other class $(L) \in I_M$ such that $(H) < (L) < (K)$.

For proper actions on a connected manifold M, Duistermaat and Kolk [7] show the existence of a unique minimal class in the isotropy lattice, say (H_0). The orbit type $M_{(H_0)}$ is called the principal orbit type and is open and dense in M.

When a proper G-action on M is not free then in general M/G is not a manifold. It is usually said that M/G is a stratified space, with the strata being the sets $M_{(H)}/G$. It is thus of crucial importance to our work to clarify the notion of stratification by orbit types and most of our work we will done in the weaker notion of a Σ-decomposition by the reasons explained below. A comprehensive reference on the subject is Pflaum [21].

Very often in the literature one encounters the stratification notion as a decomposition of a topological space into pieces (strata) that are manifolds satisfying the so-called frontier condition (if $R \cap \overline{S} \neq \emptyset$ then $R \subset \overline{S}$, for pieces R, S). As the following example from Sjamaar and Lerman [24] shows, this stratification notion is not adequate if we want to include M/G as a stratified set with strata $M_{(H)}/G$ since the set $M_{(H)}$, and consequently $M_{(H)}/G$, is not in general a manifold, but a disconnected union of manifolds of different dimensions.

Example 1. Consider the action of S^1 on $\mathbb{C}P^2$ given by

$$\mathrm{e}^{\mathrm{i}\theta} \cdot [z_0, z_1, z_2] := [\mathrm{e}^{\mathrm{i}\theta} z_0, z_1, z_2].$$

It is clear that the orbit type submanifold $M_{(S^1)}$ is then the disjoint union of the point at infinity $[1, 0, 0]$ and the complex plane $[0, z_1, z_2]$.

One could try to remedy this situation of the failure of $M_{(H)}$ to be a manifold by considering a decomposition with pieces the connected components of $M_{(H)}$. However, in this case is not clear how the frontier conditions work. For these reasons we will adopt here the notion of a Σ-decomposition.

Definition 1 (Σ-*Decomposition*). Let M be a paracompact Hausdorff space with countable topology and $\mathcal{Z}$ a locally finite partition of M into disjoint locally closed subspaces $S \subset M$. The pair $(M, \mathcal{Z})$ is called a Σ-decomposed space and $\mathcal{Z}$ a Σ-decomposition if the following conditions are satisfied:

 (i) Every piece $S \in \mathcal{Z}$ is a Σ-manifold in the induced topology, that is S is a topological sum of countably many connected smooth and separable manifolds.
(ii) If $R \cap \overline{S} \neq \emptyset$, for a pair of pieces $R, S \in \mathcal{Z}$, then $R \subset \overline{S}$ (*frontier condition*).

Σ-geometry

In general, a Σ-manifold will not be a manifold unless all its connected components have the same dimension; however, one can reproduce virtually all the geometric results traditionally stated for manifolds for these objects. In this sense, the tangent (resp. cotangent) bundle TM (resp. T^*M) of a Σ-manifold M will be the topological sum of the tangent (resp. cotangent) bundles of each connected component of M and it is naturally a Σ-manifold. A map $f : M \to N$ between Σ-manifolds is smooth if the image of the intersection of the domain of f with each connected component of M is contained in a connected component of N and the restriction of f to each connected component of M, seen as a map between connected manifolds, is smooth. This allows us to implement the concepts of diffeomorphisms, immersions, embeddings, etc. of Σ-manifolds. In the same spirit one can define vector fields, flows, group actions, etc. Because of this flexibility, many times we will simply drop the prefix Σ when these constructions arise, if the meaning is clear from the context.

The definition of a Σ-decomposition is well adapted to the decomposition of a G-manifold into orbit types. Indeed, using the Tube Theorem one can show that for a compact subgroup H of G the sets $M_{(H)}$ are locally closed Σ-submanifolds of M, meaning that each connected component of $M_{(H)}$ is a submanifold of M (for the proof see Corollary 4.2.8 and Lemma 4.2.9 of Pflaum [21]). Furthermore one can show that, for proper actions, the decomposition of M into the Σ-submanifolds $M_{(H)}$, is locally finite (see Pflaum [21] Lemma 4.3.2). We then have the following

Proposition 1. *Let M be a proper G-manifold. The orbit type decomposition of M is a Σ-decomposition with the pieces given by the orbit types $M_{(H)}$, $(H) \in I_M$. In particular, the frontier condition for the pieces becomes equivalent to*

$$M_{(H)} \cap \overline{M_{(K)}} \neq \emptyset \iff (K) \leq (H). \tag{2}$$

Notice that the larger the orbit type, the smaller the isotropy subgroup, that is $(H) \leq (K)$ if and only if $M_{(K)} \subset \overline{M}_{(H)}$.
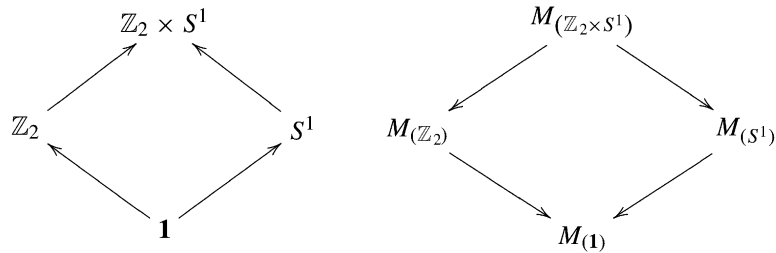
Fig. 1. Isotropy lattice and decomposition lattice for the $\mathbb{Z}_2 \times S^1$ action on $M = \mathbb{R}^3$.

An useful way to visualize the global distribution of pieces of a Σ-decomposed space M is to associate to it a decomposition lattice, where the elements are the pieces of M, together with arrows showing the frontier conditions of pairs of pieces. In this way, if R and S are two pieces we draw an arrow from R to S if $R \subset \partial S$ and there is no other piece T such that $R \subset \partial T$, and $T \subset \partial S$, where $\partial S := \overline{S} \setminus S$. For instance if our Σ-decomposition is the orbit type decomposition of a G-manifold M, we find from the previous proposition that the decomposition lattice of M has the same shape as the isotropy lattice of I_M, where in place of the representative H of an isotropy class we will have the corresponding orbit type $M_{(H)}$, and the directions of the arrows will be the reverse of those in the isotropy lattice. Sometimes these particular kinds of decomposition lattices are called orbit type lattices.

As an example consider the action of $\mathbb{Z}_2 \times S^1$ on $\mathbb{R}^3$ where S^1 acts by rotations around the x_3-axis and $\mathbb{Z}_2$ by reflections with respect to the (x_1, x_2)-plane. Since this group is compact, its resulting action on $\mathbb{R}^3$ is proper and the isotropy groups are of four types. $\mathbb{Z}_2 \times S^1$ is the stabilizer of $\mathbf{0}$, $\mathbb{Z}_2$ is the stabilizer of points of the (x_1, x_2)-plane away from the origin, S^1 is the stabilizer of points of the x_3-axis except the origin and the identity $\mathbf{1}$ is the stabilizer of the remaining points. The respective isotropy lattice and decomposition lattice are given in Fig. 1.

The Σ-decomposition of M by orbit types induces a Σ-decomposition on M/G (see for instance Theorem 4.3.10 of Pflaum [21]). Its pieces are $M_{(H)}/G$ where $H \in I_M$ (recall that by item (5) of the properties of proper group actions these spaces are Σ-manifolds) and they satisfy identical frontier conditions as the corresponding $M_{(H)}$, so the decomposition lattices of M and M/G are identical.

For further reference we define a morphism of decomposed spaces as follows.

Definition 2. A continuous map $f : (M_1, \mathcal{Z}_1) \rightarrow (M_2, \mathcal{Z}_2)$ between decomposed spaces is called a morphism of decomposed spaces if, for every piece $S \in \mathcal{Z}_1$, there is a piece $R \in \mathcal{Z}_2$ such that: (i) $f(S) \subset R$ and (ii) the restriction of f to S is smooth.

If all the restrictions $f|_S$ are injective, surjective, immersions, submersions, embeddings etc, f will be called a decomposed immersion, submersion, embedding, etc.

Finally, if $(M, \mathcal{Z}_1)$ and $(M, \mathcal{Z}_2)$ are two decompositions of the same topological space M, we say that $(M, \mathcal{Z}_1)$ is finer provided the identity map id $: (M, \mathcal{Z}_1) \rightarrow (M, \mathcal{Z}_2)$ is a morphism of decomposed spaces.

As a consequence of this definition, if S_1 and S_2 are two pieces in $\mathcal{Z}_1$ whose images under f are contained respectively in R_1 and R_2 in $\mathcal{Z}_2$ and $S_1 \subset \overline{S_2}$ then $R_1 \subset \overline{R_2}$.

2.2. Symplectic reduction at zero momentum

We now consider the setting of a Lie group G acting properly and symplectically on a symplectic manifold $\mathcal{P}$ and admitting an equivariant momentum map $\mathbf{J}$. Since 1973/74 (see [15,16]) it is known that when the action is free, one can construct reduced symplectic manifolds $\mathbf{J}^{-1}(\mu)/G_\mu$, henceforth referred to as Marsden–Weinstein (MW) reduced spaces.

When the assumption of freeness of the action is removed, the situation becomes immediately complicated as the momentum level sets are no longer in general submanifolds. Nevertheless, with the idea of partitioning the level sets into orbit types, it is possible to prove that one can obtain a symplectic stratification of the singular reduced spaces. In [24] the Marsden–Weinstein reduced space at zero momentum, $\mathcal{P}_0 = \mathbf{J}^{-1}(0)/G$, is described as a Σ-decomposition with each piece a symplectic Σ-manifold constructed using orbit types. In Theorem 2 we recall this result.

Throughout this paper we will use the following notations. Given a G-invariant subset A of a G-manifold $\mathcal{P}$ we define

$$A_{(H)} := A \cap \mathcal{P}_{(H)}, \quad \text{and} \quad A^{(H)} := A_{(H)}/G.$$

We also make use of the following subsets of a G-manifold M:

$$M_H := \{m \in M : G_m = H\}, \qquad M^H := \{m \in M : H \subset G_m\}.$$

Note that $M_{(H)} = G \cdot M_H$.

Theorem 2 (*Sjamaar and Lerman [24]*). *Let* $(\mathcal{P}, \omega)$ *be a connected symplectic manifold on which G acts properly and symplectically admitting an equivariant momentum map* $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$. *Then* $(\mathbf{J}^{-1}(0))_{(L)}$ *is a G-invariant Σ-submanifold of $\mathcal{P}$ and $\mathcal{P}_0 := \mathbf{J}^{-1}(0)/G$ is a disjoint union of smooth symplectic Σ-manifolds,*

$$\mathcal{P}_0 = \bigsqcup_{(L) \in I_{\mathcal{P}}} \mathcal{P}_0^{(L)}, \tag{3}$$

where $\mathcal{P}_0^{(L)} := (\mathbf{J}^{-1}(0))_{(L)}/G$ *with the reduced symplectic form* $\omega_0^{(L)}$ *on* $\mathcal{P}_0^{(L)}$ *given by*

$$\pi^{(L)*} \omega_0^{(L)} = i_{(L)}^* \omega,$$

where $i_{(L)} : (\mathbf{J}^{-1}(0))_{(L)} \to \mathcal{P}$ *is the inclusion, and the orbit projection is denoted by* $\pi^{(L)} : (\mathbf{J}^{-1}(0))_{(L)} \to \mathcal{P}_0^{(L)}$. *Furthermore, this partition of $\mathcal{P}_0$ is a Σ-decomposition with frontier conditions obtained from the isotropy lattice $I_{\mathcal{P}}$.*

Remark 2. In the above decomposition, some of the $\mathcal{P}_0^{(L)}$ might be empty (this happens if $\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(L)} = \emptyset$). We will refer to (3) as the *symplectic decomposition* of $\mathcal{P}_0$. ▲

In the rest of the paper we study the additional structure that the spaces $\mathcal{P}_0$ and $\mathcal{P}_0^{(L)}$ inherit from the cotangent bundle structure of the original symplectic manifold $\mathcal{P}$ extending the known classical results for the free case.

2.3. Cotangent bundle reduction

In this section we review the well known results on cotangent bundle reduction at zero momentum. We start with the free case and then we review the case of a base manifold with just one orbit type. Throughout this section we assume that G is a Lie group acting properly on a smooth manifold M and by cotangent lifts on T^*M.

The action of G on T^*M is Hamiltonian with respect to the canonical symplectic form ω and has an Ad^*-equivariant momentum map $\mathbf{J} : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(p_m), \xi \rangle = \langle p_m, \xi_M(m) \rangle, \tag{4}$$

where $p_m \in T_m^*M$ and ξ_M denotes the infinitesimal generator for the G-action on M corresponding to $\xi \in \mathfrak{g}$.

In the free case, the cotangent lifted action on T^*M is also free and proper, and consequently both orbit spaces, M/G and T^*M/G, are smooth manifolds. From (4) one has

$$\alpha_m \in \mathbf{J}^{-1}(0) \cap T_m^*M \iff \langle \alpha_m, \xi_M(m) \rangle = 0,$$

and so the zero level set of $\mathbf{J}$ is the annihilator of the bundle $V \subset TM$ defined by $V_m = \{\xi_M(m) : \xi \in \mathfrak{g}\} = T_m(G \cdot m)$. That is, $\mathbf{J}^{-1}(0) = V^0$, which is a subbundle of T^*M. The MW-reduced space $\mathcal{P}_0 := \mathbf{J}^{-1}(0)/G$ is a smooth symplectic manifold with symplectic form ω_0 induced from the canonical symplectic form ω on $\mathcal{P} = T^*M$ defined by

$$\pi^* \omega_0 = i^* \omega,$$

where $i : \mathbf{J}^{-1}(0) \to T^*M$ is the inclusion and $\pi : \mathbf{J}^{-1}(0) \to \mathcal{P}_0$ the orbit projection map. The following theorem, due to Satzer in the case of G Abelian, and Abraham and Marsden in the general case, shows that $\mathcal{P}_0$ is symplectomorphic to the cotangent bundle of the orbit space M/G, with its canonical symplectic form.

Theorem 3 (*Satzer [22], Abraham and Marsden [1]*). *If G acts freely and properly on M and by cotangent lifts on T^*M then the symplectic reduced space $(\mathcal{P}_0, \omega_0)$ is symplectomorphic to $T^*(M/G)$ equipped with its canonical symplectic structure.*

Proof. We sketch a proof as follows. Consider the map $\phi : TM/G \to T(M/G)$, defined by $\phi([v_m]) = T_m\pi(v_m)$. This map is well defined and both fiber preserving and surjective. Its dual, $\phi^* : T^*(M/G) \to T^*M/G$, is then a fiberwise injective bundle map and $\text{Im}(\phi^*) = V^0/G$. As the vector bundles $T^*(M/G)$ and V^0/G over M/G have the same dimension it follows that ϕ^* is a bundle isomorphism, i.e., $T^*(M/G) \cong V^0/G$. Finally, the symplectomorphism of the theorem is given by $(\phi^*)^{-1}$. $\quad\square$

The next easiest generalization of this result, without the freeness assumption, is the case where M consists of a single orbit type. This problem has been solved by Emmrich and Römer [8], and later by Schmah [23] with a different proof.

Theorem 4 (*Emmrich and Römer [8]*). *Let G be a Lie group acting on M properly and on $\mathcal{P} = T^*M$ by cotangent lifts. If all the points of M have isotropy groups conjugate to some $H \subset G$ (so that $M = M_{(H)}$), then $\mathbf{J}^{-1}(0) = (\mathbf{J}^{-1}(0))_{(H)}$ and $\mathcal{P}_0 = \mathcal{P}_0^{(H)} = \mathbf{J}^{-1}(0)/G$ is symplectomorphic to $T^*(M/G) = T^*M^{(H)}$ with its canonical symplectic form which we denote ω_H.*

Remark 3. The symplectomorphism of the above theorem is the same as in Theorem 3 for the free case. ▲

3. Decomposition of $\mathbf{J}^{-1}(0)$

In this section we prove a main result, Theorem 5, showing that the isotropy lattice for the G-action on $\mathbf{J}^{-1}(0)$ is identical to the isotropy lattice for the G-action on the base manifold M. This result is special for zero momentum and relies crucially on the fact that the zero momentum level set corresponds to the annihilator of the tangent spaces to the group orbits. Throughout the rest of the paper the setting will be of a Lie group G acting properly on a connected smooth manifold M and by cotangent lifts on $\mathcal{P} = T^*M$. Note that the resulting action on $\mathcal{P}$ is automatically proper.

3.1. Partition of T^*M along orbit types

Due to the properness of the action, Proposition 1 gives that M is a Σ-decomposed manifold by orbit types, that is

$$M = \bigsqcup_{(H)\in I_M} M_{(H)}, \tag{5}$$

where $M_{(H)}$ are Σ-submanifolds of M verifying the frontier condition (2).

Let $\mathbf{g}$ be a G-invariant metric on M, and use (5) to write TM as a union of Whitney sums of Σ-vector bundles, that is

$$TM = \bigsqcup_{(H)\in I_M} TM_{(H)} \oplus NM_{(H)}, \tag{6}$$

where $NM_{(H)}$ denotes the orthogonal complement to $TM_{(H)}$ as a Σ-vector bundle over $M_{(H)}$.

Since G acts by isometries, the Legendre map $\mathbb{F}L : TM \to T^*M$, defined by $\mathbb{F}L(v_m)(w_m) = \mathbf{g}(m)(v_m, w_m)$, is an equivariant bundle diffeomorphism from TM to T^*M and induces the following dual splitting

$$T^*M = \bigsqcup_{(H)\in I_M} T^*M_{(H)} \oplus N^*M_{(H)}, \tag{7}$$

which is a partition of T^*M.

Let $\mathbf{J} : T^*M \to \mathfrak{g}^*$ be the Ad^*-equivariant momentum map for the cotangent lifted action of G on T^*M. The partition (7) of T^*M along orbit types allows us to express the zero level set of the momentum map as a disjoint union of Σ-bundles over each orbit type in the base manifold.

Proposition 2. *For a proper action of G on the base manifold M the zero level set of the momentum map $\mathbf{J}$ for the lifted G-action on T^*M is a disjoint union of Σ-vector bundles over $M_{(H)}$, where (H) runs in the isotropy lattice I_M of the base manifold. In particular*

$$\mathbf{J}^{-1}(0) = \bigsqcup_{(H) \in I_M} \mathbf{J}_{(H)}^{-1}(0) \oplus N^*M_{(H)}, \tag{8}$$

where $\mathbf{J}_{(H)}$ is the momentum map for the G-action restricted to the Σ-bundle $T^*M_{(H)}$ and $N^*M_{(H)}$ is the Σ-conormal bundle of $M_{(H)}$.

Proof. Let $m \in M_{(H)}$ with stabilizer $G_m = H$. Recall that by definition of the momentum map (4) we have

$$\mathbf{J}_m^{-1}(0) = (\mathfrak{g} \cdot m)^\circ \subset T_m^*M,$$

where we use the notation $\mathbf{J}_m := \mathbf{J}|_{T_m^*M}$. We will now decompose this annihilator making use of the metric $\mathbf{g}$ and the slice construction as follows. By definition of the normal bundle $NM_{(H)}$ to the Σ-manifold $M_{(H)}$, we have

$$T_m M = T_m M_{(H)} \oplus N_m M_{(H)}. \tag{9}$$

Next, we use the metric to construct a linear slice S_m for the action of G on M at the point m,

$$T_m M = \mathfrak{g} \cdot m \oplus S_m,$$

where S_m is the orthogonal complement of the vertical space at m, i.e., $S_m = (\mathfrak{g} \cdot m)^\perp = (T_m(G \cdot m))^\perp$. We can decompose this space as follows, noting that $N_m M_{(H)}$ is orthogonal to $\mathfrak{g} \cdot m \subset T_m M_{(H)}$,

$$S_m = S_m \cap T_m M_{(H)} \oplus N_m M_{(H)}.$$

Let us denote by $S_m' := S_m \cap T_m M_{(H)}$. Note that S_m' is the orthogonal complement in $T_m M_{(H)}$ to the subspace $\mathfrak{g} \cdot m$. Therefore, by construction, it is a linear slice for the G-action restricted to the Σ-manifold $M_{(H)}$ through m. Consider the linear H-action on S_m'. Since $M_{(H)}$ has one orbit type by construction, H must fix the entire space S_m'. In fact, letting S_m^H denote the vector subspace of S_m fixed by the H action, we have $S_m^H = S_m'$. To see this, if $(a, b) \in S_m^H \oplus N_m M_{(H)}$ is fixed by H then $\exp_m |_{S_m}(a, b) \in M_H$, which implies that $(a, b) \in T_m M_{(H)}$, from which we conclude that $b = 0$. We have therefore shown that $S_m^H = S_m'$, and therefore we have the decompositions

$$T_m M = \mathfrak{g} \cdot m \oplus S_m^H \oplus N_m M_{(H)} \tag{10}$$

and

$$T_m M_{(H)} = \mathfrak{g} \cdot m \oplus S_m^H. \tag{11}$$

Taking the dual of Eq. (10) we obtain

$$T_m^* M = (S_m^H \oplus N_m M_{(H)})^\circ \oplus (\mathfrak{g} \cdot m)^\circ \simeq (\mathfrak{g} \cdot m)^* \oplus (S_m^H \oplus N_m M_{(H)})^*$$

so that $(\mathfrak{g} \cdot m)^\circ \simeq (S_m^H)^* \oplus N_m^* M_{(H)}$. Furthermore, taking the dual of Eq. (11) we obtain

$$T_m^* M_{(H)} = (S_m^H)^\circ \oplus \mathrm{Ann}(\mathfrak{g} \cdot m; T_m^* M_{(H)}) \simeq (\mathfrak{g} \cdot m)^* \oplus (S_m^H)^*$$

so that

$$\mathrm{Ann}(\mathfrak{g} \cdot m; T_m^* M_{(H)}) \simeq (S_m^H)^*. \tag{12}$$

Here we used the following notation: if $A \hookrightarrow B$ is a linear injection of vector spaces, $\mathrm{Ann}(A; B^*)$ denotes the annihilator of A in B^*. Now, since the G-action restricts to $M_{(H)}$ we can consider its cotangent lifted action to $T^*M_{(H)}$. The momentum map for this action is just the restriction of the momentum map on T^*M to $T^*M_{(H)}$. We call this momentum map $\mathbf{J}_{(H)} : T^*M_{(H)} \to \mathfrak{g}^*$. It then follows from Eq. (12) that $(S_m^H)^*$ is the zero level set of the momentum map $\mathbf{J}_{(H)}$ restricted to the fiber over $m \in M_{(H)}$. Denoting by $\mathbf{J}_{(H)m} := \mathbf{J}_{(H)}|_{T_m^* M_{(H)}}$ we have then shown that

$$\mathbf{J}_m^{-1}(0) = \mathbf{J}_{(H)m}^{-1}(0) \oplus N_m^* M_{(H)}, \tag{13}$$

from which the result follows. $\quad\square$

3.2. Orbit types of $\mathbf{J}^{-1}(0)$

In order to carry out the symplectic reduction for the zero level set $\mathbf{J}^{-1}(0)$, Theorem 2 tells us that we need to characterize $\mathcal{P}_0^{(L)} = (\mathbf{J}^{-1}(0))_{(L)}/G$, for each (L) in the isotropy lattice of the G-lifted action on T^*M.

By its very definition, the cotangent lifted action $G \times T^*M \to T^*M$ satisfies $\tau(g \cdot p_m) = g \cdot \tau(p_m)$, where the dot denotes both the left action of G on T^*M and on M, and $\tau : T^*M \to M$ denotes the projection. It is then clear that in general the isotropy lattice for the cotangent bundle, say I_{T^*M}, has more classes than I_M, although it always contains those belonging to I_M since M is G-equivariantly embedded in T^*M as the zero section. The main aim of this section is to show, in Theorem 5, that there exists a one-to-one correspondence between orbit types in M and the symplectic pieces of the reduced space $\mathcal{P}_0 = \mathbf{J}^{-1}(0)/G$. This is a remarkable feature of the zero momentum level set. We start with the following coarse description of $\mathbf{J}^{-1}(0)$ which will be refined in the subsequent theorem.

Proposition 3. *The orbit types of the zero level set of the momentum map for the cotangent lifted action of G on T^*M are expressed as*

$$(\mathbf{J}^{-1}(0))_{(L)} = \bigsqcup_{(H) \geq (L)} \mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)}, \qquad (14)$$

where (H) is in I_M and (L) is fixed in I_{T^*M}.

Proof. As the projection $\tau : T^*M \to M$ is equivariant and $(L) \in I_{T^*M}$, then $\tau((\mathbf{J}^{-1}(0))_{(L)}) \cap M_{(H)} \neq \emptyset$ implies $(L) \leq (H)$. So, from (8) we get

$$(\mathbf{J}^{-1}(0))_{(L)} = \bigsqcup_{(H) \geq (L)} (\mathbf{J}_{(H)}^{-1}(0) \oplus N^*M_{(H)})_{(L)}.$$

Recall that $\mathbf{J}_{(H)}$ is the momentum map for the cotangent lifted G-action to $T^*M_{(H)}$. We can now apply the single orbit type theorem for cotangent lifted actions (Theorem 4) to obtain $\mathbf{J}_{(H)}^{-1}(0) = (\mathbf{J}_{(H)}^{-1}(0))_{(H)}$, which gives the result.   □

At this point, we are able to get more information on the possible subgroups (L) by a careful analysis of the G-action on the conormal bundles $N^*M_{(H)}$. The key to getting finer information is to apply the slice construction and the Tube Theorem both for the G-action on $M_{(H)}$ and for the G-action on M. This will allow us to relate the orbit types for the G-action on the conormal bundle to the orbit types for the G-action on the base. Specifically, we find,

Theorem 5. *For any $m \in M_{(H)}$ such that $G_m = H$, and any fixed $(L) \in I_{T^*M}$, then the orbit type (L) of the zero level set of the momentum map for the lifted G-action, restricted to the fiber over m, verifies*

$$(\mathbf{J}^{-1}(0))_{(L)} \cap T_m^*M \neq \emptyset$$

if and only if both of the following conditions hold:

(i) $(L) \leq (H)$,   (ii) $M_{(L)} \neq \emptyset$.

Before proving Theorem 5 we will prove a lemma relating the orbit types for the linear action of a subgroup H of G on $S_m = (\mathfrak{g} \cdot m)^\perp$ and the orbit types of G on the base manifold M. It seems that most of the results in this lemma are scattered in the literature in a different form and so we present here a version that is better adapted to our purposes.

Lemma 1. *Let $m \in M_{(H)}$ with $G_m = H$, B_r a ball of radius r around zero in $S_m = (\mathfrak{g} \cdot m)^\perp$ with r smaller than the injectivity radius of $\exp_m$, U and ϕ respectively the G-invariant neighborhood of $G \cdot m$ and the diffeomorphism given by the Tube Theorem and $M_{(K)} \neq \emptyset$ for some $(K) \in I_M$. Consider the linear H action on S_m. Then:*

1. *$U \cap M_{(K)} \neq \emptyset$ if and only if $(K) \leq (H)$.*
2. *$(S_m)_{(L)} \neq \emptyset$, if and only if there exists a class $(K) \in I_M$ with $(K) \leq (H)$ such that L is conjugate in G to a representative of (K).*
3. *The set of points $[G, u] \subset G \times_H B_r$ with $u \in (B_r)_{(L)}$ gets mapped by ϕ into $M_{(K)}$, where K is a subgroup of H conjugate in G to L.*

Proof. 1: Suppose $(K) \leq (H)$, then by the frontier condition we have $M_{(H)} \subset \overline{M_{(K)}}$. So, every open set in M containing a point in $M_{(H)}$ must have nonempty intersection with $M_{(K)}$.

Conversely, suppose $m' \in U \cap M_{(K)}$. Then $G_{m'}$ is conjugate to K in G, i.e., $(G_{m'}) = (K)$. On the other hand, $m' \in U$ and $U = G \cdot \exp_m(B_r)$, so $m' = g \cdot s$ for some $s \in \exp_m(B_r) \subset M$ and $g \in G$. Thus, as $m' = gs$ then $(G_{m'}) = (G_s)$ and as $s \in \exp_m(B_r)$ then [Remark 1](#) gives $G_s \subseteq H$. So $(G_{m'}) = (G_s) = (K) \leq (H)$.

For 2: From 1, we know that for $(K) \leq (H)$ there exists $s \in \exp_m(B_r)$ such that $L := G_s$ is conjugate to K. Since $\exp_m$ is H-equivariant, the point $\exp_m^{-1}(s) \cap S_m \in B_r$ is stabilized by L under the linear H action on B_r. Since this action extends linearly to the entire space S_m, we conclude that $(S_m)_{(L)} \neq \emptyset$, where L is conjugate to K. Conversely, let L be a subgroup of H such that $(S_m)_{(L)} \neq \emptyset$. By linearity of the H action, $(B_r)_{(L)} \neq \emptyset$, and by equivariance of $\exp_m$, we have $(\exp_m(B_r))_{(L)} \neq \emptyset$. By 1, this immediately implies that L is conjugate to K for some $(K) \in I_M$ with $(K) \leq (H)$.

Finally, to prove 3, it is sufficient to take $u \in (B_r)_{(L)}$ so that $H_u = L$. Now, since $\exp_m$ is H equivariant, we have that $\exp_m(u)$ is stabilized by L as well, and in fact $G_{\exp_m(u)} = L$. It follows that each point in the set $\phi([G, u]) = G \cdot \exp(u)$ is contained in $M_{(L)} = M_{(K)}$ as required. $\square$

Proof (*Of [Theorem 5](#)*). Recall from the proof of [Proposition 2](#) that

$$\mathbf{J}_m^{-1}(0) = (\mathfrak{g} \cdot m)^\circ \simeq S_m^* = (S_m^H \oplus N_m M_{(H)})^* \simeq (S_m^H)^* \oplus N_m^* M_{(H)}.$$

Since H acts by isometries on $T_m M$ and on $T_m M_{(H)}$ by restriction then H maps $N_m M_{(H)}$ into itself and the action of H on $S_m^H \oplus N_m M_{(H)}$ is diagonal. Furthermore the H action on S_m^H is trivial by construction.

Therefore for $(a, b) \in S_m^H \times N_m M_{(H)}$ one has $H_{(a,b)} = H_b$, as $H_{(a,0)} = H$. Consequently, the orbit type sets for the H action on S_m are of the form $S_m^H \times (N_m M_{(H)})_{(L)}$, where (L) belongs to the isotropy lattice for the linear H action on S_m.

Let us show that if $b \neq 0$ then H_b is strictly contained in H. For this, note that locally S_m^H and S_m are linear slices at m for the G-actions on $M_{(H)}$ and M respectively.

Consider the direct product of H-invariant neighborhoods $B_{r_1} \times B_{r_2} \subset S_m^H \oplus N_m M_{(H)}$, each of them inside the disk of radius $r_i > 0$ centered at 0 in the corresponding vector space, where $r_1^2 + r_2^2 < r^2$. Then, their direct product is contained in the disk $B_r \subset S_m$. Denote by $\phi_{M_{(H)}} : G \times_H B_{r_1} \to U_{M_{(H)}}$ the diffeomorphism from [Theorem 1](#) applied to the slice for the G-action on $M_{(H)}$. The image of $\phi_{M_{(H)}}$ is an open G-invariant set of $M_{(H)}$ and not of M. Next consider the slice at the point m for the entire manifold M, modelled on the space $G \times_H (B_{r_1} \times B_{r_2})$, and the corresponding map $\phi : G \times_H (B_{r_1} \times B_{r_2}) \to U$. Suppose there exists $0 \neq y \in B_{r_2}$ such that $H_y = H$. Then, the entire open set $B_{r_1} \times ty$ where $t \in (0, r_2/\|y\|)$ is stabilized by H and therefore, by (3) of [Lemma 1](#), $\phi([G \times_H (B_{r_1} \times ty)])$ is contained in $M_{(H)}$. However, ϕ is a diffeomorphism so this image has one higher dimension than $\phi_{M_{(H)}}(G \times_H B_{r_1})$. On the other hand, they are both open sets in $M_{(H)}$, which is a contradiction. We have then proved that $H_b \subsetneq H$ for $b \neq 0$.

From (2) of [Lemma 1](#) we know that $(S_m)_{(L)} \neq \emptyset$ if and only if L is conjugate to $K \subseteq H$ for some $(K) \in I_M$ and $M_{(K)} \neq \emptyset$. Then we have proved that

$$(\mathbf{J}_m^{-1}(0))_{(L)} = S_m^H \times (N_m^* M_{(H)})_{(L)} \neq \emptyset$$

if and only if

$$(L) \leq (H) \quad \text{and} \quad M_{(L)} \neq \emptyset. \quad \square$$

From the proof of [Theorem 5](#) and noting that $M_{(H)} = G \cdot M_H$ we have

Corollary 6. $(N^* M_{(H)})_{(L)} \neq \emptyset$ *if and only if* $(H) \geq (L)$ *and* $M_{(L)} \neq \emptyset$. *Furthermore* $(N^* M_{(H)})_{(H)}$ *is the zero section of the Σ-bundle* $N^* M_{(H)} \to M_{(H)}$, *i.e., it is isomorphic to* $M_{(H)}$.

To end this section we summarize in the next proposition the main results obtained so far for the orbit types of the zero momentum level set.

Proposition 4. *In the previous conditions we have:*

(a) $(L) \in I_{\mathbf{J}^{-1}(0)} \iff (L) \in I_M$ *and then* $\mathcal{P}_0^{(L)} \neq \emptyset \iff (L) \in I_M$.

(b) *The cotangent bundle projection τ restricts to the G-equivariant continuous surjection* $\tau_L : (\mathbf{J}^{-1}(0))_{(L)} \to \overline{M_{(L)}}$.

(c) *A fixed orbit type* (L) *in the zero momentum level set is a* Σ*-submanifold of* T^*M *which admits the following* G*-invariant partition:*

$$(\mathbf{J}^{-1}(0))_{(L)} = \mathbf{J}_{(L)}^{-1}(0) \bigsqcup_{(H)>(L)} \mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)}. \tag{15}$$

(d) *For every* $(H) > (L)$*, the restrictions*

$$t_L := \tau_L|_{\mathbf{J}_{(L)}^{-1}(0)} \quad and \quad t_{H \to L} := \tau_L|_{\mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)}}$$

are G*-equivariant smooth surjective submersions respectively onto* $M_{(L)}$ *and* $M_{(H)}$.

Proof. Statement (a) is proved in Theorem 5. For (b): To prove continuity of τ_L, first note that $\overline{M_{(L)}}$ has the relative topology from M so we must show that for any open set U in M, $\tau_L^{-1}(U \cap \overline{M_{(L)}})$ is open in $(\mathbf{J}^{-1}(0))_{(L)}$. The cotangent projection $\tau : T^*M \to M$ is of course continuous, so $\tau^{-1}(U)$ is open in T^*M and therefore $\tau^{-1}(U) \cap (\mathbf{J}^{-1}(0))_{(L)}$ is an open set in $(\mathbf{J}^{-1}(0))_{(L)}$. It is easy to show that $\tau^{-1}(U) \cap (\mathbf{J}^{-1}(0))_{(L)} = \tau_L^{-1}(U \cap \overline{M_{(L)}})$, from which continuity of τ_L follows. G-equivariance is obvious. Finally, note that the image of τ restricted to $(\mathbf{J}^{-1}(0))_{(L)}$ is the disjoint union $\bigsqcup_{(H) \geq (L)} M_{(H)} = \overline{M_{(L)}}$ since for each $(H) \geq (L)$, $\mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)}$ is a Σ-fiber bundle over $M_{(H)}$. (c) just follows from Proposition 3 and Theorem 5. To obtain (d), note that $\mathbf{J}_{(L)}^{-1}(0)$ is a Σ-fiber bundle over $M_{(L)}$, i.e., disjoint union of smooth fiber bundles over each connected component of $M_{(L)}$ and on each connected component the fiber bundle projection t_L is a smooth surjective submersion. G-equivariance follows from the definition of the cotangent lifted action. Similarly $\mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)}$ is a Σ-fiber bundle over $M_{(H)}$ with smooth surjective Σ-submersion $t_{H \to L}$. $\square$

4. Topology and symplectic geometry of $\mathcal{P}_0$

The general symplectic reduction theory (Theorem 2) tells us that $\mathcal{P}_0$ is a Σ-decomposed space with symplectic pieces $\mathcal{P}_0^{(L)}$. In the specific case of a cotangent bundle, we show in this section that these symplectic pieces also admit a Σ-decomposition which we call the *secondary decomposition*. The pieces of the secondary decomposition of $\mathcal{P}_0^{(L)}$ are studied in detail and we are able to prove that there exists an open and dense piece which is diffeomorphic to the cotangent bundle of $M_{(L)}/G$. The other pieces will be called seams.

The reduced symplectic data then have a natural interpretation. The reduced symplectic form $\omega_0^{(L)}$ in the symplectic piece $\mathcal{P}_0^{(L)}$ can be obtained as the unique smooth extension from this open dense part of the canonical symplectic form on $T^*(M_{(L)}/G)$. Relative to the reduced symplectic forms we will prove that the seams are coisotropic Σ-submanifolds of $(\mathcal{P}_0^{(L)}, \omega_0^{(L)})$.

We already know that the reduced space at zero momentum $\mathcal{P}_0$, admits a symplectic Σ-decomposition in symplectic pieces (Theorem 2). We will prove that, joining together all the pieces of the secondary decomposition of each symplectic piece $\mathcal{P}_0^{(L)}$, the resulting partition of $\mathcal{P}_0$ is another Σ-decomposition, which we call the *coisotropic decomposition*. We explicitly identify the frontier conditions for both Σ-decompositions of $\mathcal{P}_0$ and $\mathcal{P}_0^{(L)}$ and show that the referred seams play a "stitching role", i.e., they stitch the cotangent bundles appearing in the coisotropic decomposition of $\mathcal{P}_0$, as we shall show in Theorem 10.

4.1. The secondary decomposition of $\mathcal{P}_0^{(L)}$

We introduce the following notation. Recall that a connectable pair $H \to L$ is a pair of elements $(H), (L) \in I_M$ such that $(H) \geq (L)$. Define the following fiber bundles

$$s_{H \to L} := \mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)} \to M_{(H)}. \tag{16}$$

where the index $H \to L$ runs over the set of connectable pairs over a fixed isotropy class (L). As this is a G-invariant piece in the G-invariant partition (15) of $(\mathbf{J}^{-1}(0))_{(L)}$, we can quotient by the G-action to obtain

$$S_{H \to L} := \pi^{H \to L}(s_{H \to L}) = \frac{\mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)}}{G} \tag{17}$$

where $\pi^{H \to L} := \pi^{(L)}|_{s_{H \to L}}$. We shall then call $S_{H \to L}$, which is a fiber bundle over $M_{(H)}/G$, a *seam* from H to L, and $s_{H \to L}$, the fiber bundle over $M_{(H)}$, a *pre-seam*.

We then have the following partition of $\mathcal{P}_0^{(L)} = (\mathbf{J}^{-1}(0))_{(L)}/G$:

$$\mathcal{P}_0^{(L)} = \mathbf{J}_{(L)}^{-1}(0)/G \bigsqcup_{(H) > (L)} S_{H \to L}. \tag{18}$$

Note that from Proposition 4(a) the conjugacy classes (L) and (H) appearing in the above equations belong to I_M, with (L) fixed in the disjoint union. Moreover, due to the G-equivariance of the restrictions of the cotangent bundle projection, referred to in (b) and (d) of Proposition 4, we have

(i) The map τ_L descends to a continuous surjection, say $\tau^L : \mathcal{P}_0^{(L)} \to \overline{M^{(L)}}$, where $\overline{M^{(L)}}$ is the closure of $M^{(L)} = M_{(L)}/G$.

(ii) For every $(H) > (L)$, the maps t_L and $t_{H \to L}$ of Proposition 4(d) descend to the following surjective submersions:

$$t^L : \mathbf{J}_{(L)}^{-1}(0)/G \to M^{(L)} \qquad t^{H \to L} : S_{H \to L} \to M^{(H)}.$$

These maps are summarized in the following commutative diagrams.

$$
\begin{array}{ccc}
\mathbf{J}_{(L)}^{-1}(0)/G & \xhookrightarrow{\;i_0^L\;} & \mathcal{P}_0^{(L)} \\
{\scriptstyle t^L}\big\downarrow & & \big\downarrow{\scriptstyle \tau^L} \\
M^{(L)} & \xhookrightarrow{\;i^L\;} & \overline{M^{(L)}}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
S_{H \to L} & \xhookrightarrow{\;i_0^{H \to L}\;} & \mathcal{P}_0^{(L)} \\
{\scriptstyle t^{H \to L}}\big\downarrow & & \big\downarrow{\scriptstyle \tau^L} \\
M^{(H)} & \xhookrightarrow{\;i^H\;} & \overline{M^{(L)}}
\end{array}
$$

Note that we know, from the general symplectic reduction theory, that $\mathcal{P}_0^{(L)}$ is a smooth (symplectic) Σ-manifold, but, recalling that $M^{(L)} := M_{(L)}/G$, $\overline{M^{(L)}}$ in general is only a topological space, endowed with the relative topology of M/G. In the next proposition we show that $\overline{M^{(L)}}$ is a Σ-decomposed space and we identify the frontier conditions for its pieces.

Proposition 5. $\overline{M^{(L)}}$ *is a Σ-decomposed space with pieces $M^{(H)}$, for all $(H) \geq (L)$. The frontier conditions are given by*

$$M^{(K)} \cap \overline{M^{(H)}} \neq \emptyset \iff (K) \geq (H).$$

Furthermore $M^{(L)}$ is open and dense in $\overline{M^{(L)}}$.

Proof. Using that $\overline{M_{(L)}} = \bigsqcup_{(H) \geq (L)} M_{(H)}$ and

$$\overline{M^{(L)}} = \bigsqcup_{(H) \geq (L)} M_{(H)}/G = \bigsqcup_{(H) \geq (L)} M^{(H)}.$$

Since the orbit type decomposition of M is a Σ-decomposition with pieces $M_{(H)}$, for all $(H) \in I_M$, it is easy to see that $\overline{M_{(L)}}$ is also a Σ-decomposed space with pieces $M_{(H)}$ with $(H) \in I_M$ and $(H) \geq (L)$. Since an orbit type decomposition of M induces a Σ-decomposition of M/G with pieces $M_{(H)}/G$ then, by the same argument as before, $\overline{M^{(L)}}$ is a Σ-decomposed space with the obvious frontier conditions stated in the proposition.

Therefore it remains to prove that $M^{(L)}$ is open and dense in $\overline{M^{(L)}}$. Density is obvious. For openness, consider a point $x \in M^{(L)} = M_{(L)}/G$ and an open neighborhood U' of x in $\overline{M^{(L)}}$. This means that there exists an open neighborhood U of x in M/G with $U' = U \cap \overline{M^{(L)}}$. Adjusting U we can assure that $U \cap M^{(H)} = \emptyset$ for every $(H) > (L)$, since the points that are stabilized by (H) lie in the boundary of $M_{(L)}$. For such a U then, $U' = U \cap \overline{M^{(L)}}$ is totally contained in $M^{(L)}$. $\square$

The element $\mathbf{J}_{(L)}^{-1}(0)/G$ of the partition (18) of $\mathcal{P}_0^{(L)}$ is diffeomorphic to the cotangent bundle of $M^{(L)}$ by the single orbit type theorem (Theorem 4), since $\mathbf{J}_{(L)}$ is the momentum map for the restriction of the G-action to $T^*M_{(L)}$. We will denote this piece by C_L, and the partition (18) can be written as

$$\mathcal{P}_0^{(L)} = C_L \bigsqcup_{(H)>(L)} S_{H\to L},\tag{19}$$

for all $(L), (H) \in I_M$. Note also that the piece C_L of the partition (19), which is diffeomorphic to a cotangent bundle, can also be seen as a seam from L to L since, by Corollary 6, $(N^*M_{(L)})_{(L)}$ is the zero section of the Σ-bundle $N^*M_{(L)} \to M_{(L)}$ and so Definition (17) gives

$$C_L = S_{L\to L} \simeq \mathbf{J}_{(L)}^{-1}(0)/G \simeq T^*M^{(L)}.$$

If there is no danger of confusion we will use $S_{L\to L}$, C_L and $T^*M^{(L)}$ to denote the same piece. Before stating the main result of this subsection we need to prove the openness of the surjective map τ_L given in Proposition 4(b).

Lemma 2. *The map $\tau_L : (\mathbf{J}^{-1}(0))_{(L)} \to \overline{M_{(L)}}$ is an open map. In addition, the quotient map, $\tau^L : \mathcal{P}_0^{(L)} \to \overline{M_{(L)}}/G$, is also open.*

Proof. We begin by considering, for a fixed $(H) \geq (L)$, $s_{H\to L} := \mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)} \hookrightarrow T^*M|_{M_{(H)}} \hookrightarrow T^*M$. The above sequence is then a sequence of embedded Σ-submanifolds. Furthermore, the pre-seam $s_{H\to L}$ is a Σ-fiber bundle over $M_{(H)}$ which embeds as a Σ-fiber subbundle of the Σ-vector bundle $T^*M|_{M_{(H)}}$. Since the topology of $(\mathbf{J}^{-1}(0))_{(L)}$ and $s_{H\to L}$ for each $(H) \geq (L)$ is the relative topology of a Σ-submanifold of T^*M, the open sets of $(\mathbf{J}^{-1}(0))_{(L)}$ are $(\mathbf{J}^{-1}(0))_{(L)} \cap U$ for each open set U in T^*M. To prove the openness of the map τ_L we need to show that $\tau_L(\mathbf{J}^{-1}(0)_{(L)} \cap U)$ is an open set in $\overline{M_{(L)}}$. Now, since

$$\tau_L((\mathbf{J}^{-1}(0))_{(L)} \cap U) = \tau_L \left(\bigsqcup_{(H)\geq(L)} s_{H\to L} \cap U \right)$$
$$= \bigsqcup_{(H)\geq(L)} t_{H\to L}(s_{H\to L} \cap U),\tag{20}$$

we need to consider the sets $t_{H\to L}(s_{H\to L} \cap U)$ contained in $M_{(H)}$. In fact we will establish the following intersection formula for an arbitrary open set $U \subset T^*M$,

$$t_{H\to L}(U \cap s_{H\to L}) = \tau(U) \cap M_{(H)},\tag{21}$$

from which the proof of openness will be an easy consequence. Abstracting slightly, given an embedding of fiber bundles, where the embeddings are inclusions,

$$\begin{array}{ccc} A_1 & \lhook\joinrel\longrightarrow & A_2 \\ {\scriptstyle\tau_1}\downarrow & & \downarrow{\scriptstyle\tau_2} \\ M_1 & \lhook\joinrel\longrightarrow & M_2 \end{array}$$

and given an open set U in A_2, it is a general result that

$$\tau_2(U) \cap M_1 = \tau_1(U \cap A_1).$$

Notice that since the fiber projection maps τ_1 and τ_2 are surjective submersions, they are open maps and therefore the left-hand side of the previous equation is open in M_1 since its open sets are generated from the relative topology and $\tau_2(U)$ is an open set in M_2. Similarly the right-hand side is also an open set in M_1. Note that this result also holds for a Σ-fiber bundle embedding. Applying this result to the Σ-fiber bundle $s_{H\to L} \hookrightarrow T^*M$ which fibers over the base inclusion $M_{(H)} \hookrightarrow M$, we conclude that the intersection formula (Eq. (21)) holds and therefore, following Eq. (20), we have

$$\tau_L((\mathbf{J}^{-1}(0))_{(L)} \cap U) = \bigsqcup_{(H)\geq(L)} t_{H\to L}(s_{H\to L} \cap U) = \bigsqcup_{(H)\geq(L)} \tau(U) \cap M_{(H)}$$
$$= \tau(U) \cap \bigsqcup_{(H)\geq(L)} M_{(H)} = \tau(U) \cap \overline{M_{(L)}}.$$

However, $\tau(U) \cap \overline{M_{(L)}}$ is an open set in $\overline{M_{(L)}}$ since $\tau(U)$ is open in M and $\overline{M_{(L)}}$ has the relative topology from M.

Next we consider the map τ^L defined through the G-equivariance of the map τ_L giving the following commutative diagram.

$$
\begin{array}{ccc}
(\mathbf{J}^{-1}(0))_{(L)} & \xrightarrow{\ \tau_L\ } & \overline{M_{(L)}} \\
\pi^{(L)} \downarrow & & \downarrow \pi_M^{(L)} \\
\mathcal{P}_0^{(L)} & \xrightarrow[\ \tau^L\ ]{} & \overline{M^{(L)}}
\end{array}
$$

The vertical arrows in this diagram are open maps since they are quotients of a G-action and the topology on the base is given by the quotient topology. Therefore, by openness of the map τ_L, given an open set U in $\mathcal{P}_0^{(L)}$, the set $\tau_L((\pi^{(L)})^{-1}(U))$ is open in $\overline{M_{(L)}}$, and therefore since

$$
\tau^L(U) = \pi_M^{(L)}(\tau_L((\pi^{(L)})^{-1}(U)))
$$

we conclude, by openness of the map $\pi_M^{(L)}$, that $\tau^L(U)$ is open. $\quad\square$

We are now able to prove one of the main results of this section.

Theorem 7. *The partition* (19) *is a Σ-decomposition of $\mathcal{P}_0^{(L)}$ that will be called the secondary decomposition of $\mathcal{P}_0^{(L)}$. The piece C_L is open and dense and diffeomorphic to $T^*M^{(L)} = T^*(M_{(L)}/G)$. The frontier conditions are:*

(1) $S_{H\to L} \subset \partial\, C_L$ *for all* $(H) > (L)$.
(2) $S_{H'\to L} \subset \partial S_{H\to L}$ *if and only if* $(H') > (H) > (L)$.

The map $\tau^L : \mathcal{P}_0^{(L)} \to \overline{M^{(L)}}$ is a Σ-decomposed surjective submersion.

Proof. By construction of (19) and because an orbit type decomposition is a Σ-decomposition it is then clear that the partition (19) is a locally finite partition. Since the pieces of the partition are Σ-submanifolds of $\mathcal{P}_0^{(L)}$ then they are automatically locally closed.

Let us prove that C_L is open and dense. Let U be an open neighborhood of $z \in \mathcal{P}_0^{(L)} = (\mathbf{J}^{-1}(0))_{(L)}/G$. Since, by Lemma 2, the map $\tau^L : \mathcal{P}_0^{(L)} \to \overline{M^{(L)}}$ is open, $\tau^L(U) = O$ is an open set in $\overline{M^{(L)}}$. By Proposition 5, $M^{(L)}$ is dense in $\overline{M^{(L)}}$ and so $O \cap M^{(L)} \neq \emptyset$. For $y \in O \cap M^{(L)}$, we have $(\tau^L)^{-1}(y) = (t^L)^{-1}(y) \subset C_L$ and $(\tau^L)^{-1}(y) \cap U \neq \emptyset$. It follows that $U \cap C_L \neq \emptyset$, proving the density. For the openness of C_L note that, by Proposition 5, $M^{(L)}$ is open and so $(\tau^L)^{-1}(M^{(L)}) = C_L$ is also open by the continuity of τ^L.

For (1), let $z \in S_{H\to L}$ with $(L) < (H)$ and U an open neighborhood of z. As C_L is dense in $\mathcal{P}_0^{(L)}$ then $U \cap C_L \neq \emptyset$. Furthermore, as C_L and $S_{H\to L}$ are disjoint for $(L) < (H)$ it follows that $z \in \partial C_L$.

Let us now prove (2). By the openness property of τ^L then any neighborhood U of a point $z \in S_{H'\to L}$ in $\mathcal{P}_0^{(L)}$ is mapped by τ^L to an open neighborhood of $\tau^L(z)$ in $\overline{M^{(L)}}$, say O. Then $O \cap M^{(H)} \neq \emptyset$ if and only if $(H') > (H) > (L)$ because $\overline{M^{(L)}}$ is a Σ-decomposed space. Then for $y \in O \cap M^{(H)}$ we have $(t^{H\to L})^{-1}(y) \cap U \neq \emptyset$, proving (2).

The map τ^L restricted to each seam is a surjective submersion, that is $\tau^L(S_{H'\to L}) = t^{H'\to L}(S_{H'\to L}) = M^{(H')}$; also $\tau^L(S_{H\to L}) = t^{H\to L}(S_{H\to L}) = M^{(H)}$. By the frontier conditions we get that τ^L is a Σ-decomposed surjective submersion. $\quad\square$

We will now describe the symplectic structure of the symplectic pieces $\mathcal{P}_0^{(L)}$. Recall that by the single orbit type theorem (Theorem 4), for each $(H) \in I_M$ there is a diffeomorphism

$$
\psi^H : C_H \to T^*M^{(H)} \tag{22}
$$

which is a Σ-bundle map covering the identity in $M^{(H)}$. Consider now, for each piece in the partition (15) of $(\mathbf{J}^{-1}(0))_{(L)}$, the projection,

$$
p_{1\,H\to L} : \mathbf{J}_{(H)}^{-1}(0) \times (N^*M_{(H)})_{(L)} \to \mathbf{J}_{(H)}^{-1}(0).
$$

Notice that this map is just the identity map on the first element of the partition, $\mathbf{J}_{(L)}^{-1}(0)$. These are equivariant maps that descend to surjective submersions

$$p_1^{H \to L} : S_{H \to L} \to C_H = \mathbf{J}_{(H)}^{-1}(0)/G. \tag{23}$$

Then for any connectable pair $H \to L$ over (L), we have for the corresponding piece $S_{H \to L}$ of $\mathcal{P}_0^{(L)}$, a surjective submersion

$$\overline{\psi}^{H \to L} = \psi^H \circ p_1^{H \to L} : S_{H \to L} \to T^*M^{(H)} \tag{24}$$

covering the identity on $M^{(H)}$. In the particular case $(H) = (L)$ we have that $S_{L \to L} = C_L$ and $\overline{\psi}^{L \to L} = \psi^L$ is a diffeomorphism. If we denote by ω_H the canonical symplectic form in $T^*M^{(H)}$ we can then induce on each piece of the secondary decomposition of $\mathcal{P}_0^{(L)}$ a closed two form by

$$\text{on } \quad C_L : \quad \Omega_L := \psi^{L*}\omega_L, \quad \text{and} \quad \text{on} \quad S_{H \to L} : \quad \Lambda_{H \to L} := \overline{\psi}^{H \to L*}\omega_H.$$

Then Ω_L is symplectic and $\Lambda_{H \to L}$ is degenerate.

By Theorem 2 the piece $\mathcal{P}_0^{(L)}$ has an abstractly defined reduced symplectic form $\omega_0^{(L)}$. It is then natural to ask to what extent the structures introduced so far are compatible. The answer to this question is given in the next proposition, which together with Theorem 8 are the main results characterizing the symplectic geometry of $\mathcal{P}_0^{(L)}$.

Proposition 6. *Let $T^*M^{(H)}$ be equipped with the canonical symplectic form ω_H and $\mathcal{P}_0^{(L)}$ with the symplectic form $\omega_0^{(L)}$ given by Theorem 2. Let also $\overline{\psi}^{H \to L}$ and ψ^L be respectively the surjective submersion (24) and the diffeomorphism (22). Then, there are closed two-forms, Ω_L on C_L and $\Lambda_{H \to L}$ on $S_{H \to L}$ defined by*

$$\Omega_L := \psi^{L*}\omega_L, \qquad \Lambda_{H \to L} := (\overline{\psi}^{H \to L})^*\omega_H,$$

verifying

$$\text{(i)} \quad \omega_0^{(L)}|_{C_L} = \Omega_L \qquad \text{and} \qquad \text{(ii)} \quad \omega_0^{(L)}|_{S_{H \to L}} = \Lambda_{H \to L}.$$

Proof. We will present the proof for (ii) from which (i) follows by taking $(H) = (L)$ and noting that $\overline{\psi}^{H \to L} = \psi^L$. First note that by Theorem 2 the symplectic form, $\omega_0^{(L)}$, in $\mathcal{P}_0^{(L)}$ is given by

$$\pi^{(L)*}\omega_0^{(L)} = i_{(L)}^*\omega, \tag{25}$$

where ω is the canonical symplectic form in T^*M, $\pi^{(L)}$ and $i_{(L)}$ respectively the orbit projection and the inclusion defined in the referred theorem (see also the diagram below). In order to prove equation (ii) let us consider the following diagram:

$$
\begin{array}{ccccc}
s_{H \to L} & \xrightarrow{\ i_{H \to L}\ } & (\mathbf{J}^{-1}(0))_{(L)} & \xrightarrow{\ i_{(L)}\ } & T^*M \\
& \downarrow{\scriptstyle \pi^{H \to L}} & & \downarrow{\scriptstyle \pi^{(L)}} & \\
T^*M^{(H)} \xleftarrow{\ \overline{\psi}^{H \to L}\ } S_{H \to L} & \xrightarrow{\ i_0^{H \to L}\ } & \mathcal{P}_0^{(L)} & &
\end{array}
$$

As $\pi^{H \to L}$ is a submersion, if we prove

$$(\pi^{H \to L})^*(i_0^{H \to L})^*\omega_0^{(L)} = (\pi^{H \to L})^*(\overline{\psi}^{H \to L})^*\omega_H, \tag{26}$$

the claim $(i_0^{H \to L})^*\omega_0^{(L)} = (\overline{\psi}^{H \to L})^*\omega_H$ of the proposition follows.

From the above diagram we have $i_0^{H \to L} \circ \pi^{H \to L} = \pi^{(L)} \circ i_{H \to L}$. So the left-hand side of (26) becomes

$$(\pi^{H \to L})^*(i_0^{H \to L})^*\omega_0^{(L)} = i_{H \to L}^*\pi^{(L)*}\omega_0^{(L)} = i_{H \to L}^*i_{(L)}^*\omega, \tag{27}$$

where the second identity follows from the definition (25) of $\omega_0^{(L)}$.

Note that the image of $i_{(L)} \circ i_{H \to L}$ is contained in $T^*M|_{M_{(H)}} \subset T^*M$. Therefore, denoting by ϕ and i_H the following inclusions

$$s_{H \to L} \overset{\phi}{\hookrightarrow} T^*M|_{M_{(H)}} \overset{i_H}{\hookrightarrow} T^*M,$$

Eq. (27) is equivalent to

$$(\pi^{H \to L})^* (i_0^{H \to L})^* \omega_0^{(L)} = i_{H \to L}^* \, i_{(L)}^* \omega = \phi^* \, i_H^* \, \omega. \tag{28}$$

So in order to prove (26) it remains to show that

$$\phi^* \, i_H^* \, \omega = (\pi^{H \to L})^* (\overline{\psi}^{H \to L})^* \omega_H. \tag{29}$$

For (29) recall that $\overline{\psi}^{H \to L} = \psi^H \circ \mathrm{p}_1^{H \to L}$. Then the right-hand side of (29) is given by

$$
\begin{aligned}
(\pi^{H \to L})^* (\overline{\psi}^{H \to L})^* \omega_H &= (\pi^{H \to L})^* (\mathrm{p}_1^{H \to L})^* (\psi^H)^* \omega_H \\
&= (\mathrm{p}_{1_{H \to L}})^* (\pi^H)^* (\psi^H)^* \omega_H
\end{aligned} \tag{30}
$$

where the second identity follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
s_{H \to L} & \overset{\mathrm{p}_{1_{H \to L}}}{\longrightarrow} & \mathbf{J}_{(H)}^{-1}(0) \\
{\scriptstyle \pi^{H \to L}} \downarrow & & \downarrow {\scriptstyle \pi^H} \\
S_{H \to L} & \overset{\mathrm{p}_1^{H \to L}}{\longrightarrow} & \mathbf{J}_{(H)}^{-1}(0)/G
\end{array}
$$

Recall that $\mathbf{J}_{(H)}$ is the momentum map for the G-action on $T^*M_{(H)}$ and so by the single orbit type theorem (Theorem 4), $\mathbf{J}_{(H)}^{-1}(0)/G$ is symplectic with symplectic form, $(\psi^H)^* \omega_H$, induced from the canonical symplectic form $\omega_{(H)}$ on $T^*M_{(H)}$, given by

$$(\pi^H)^* (\psi^H)^* \omega_H = j^* \omega_{(H)}, \tag{31}$$

where j denotes the inclusion $j : \mathbf{J}_{(H)}^{-1}(0) \to T^*M_{(H)}$.

Using Eq. (31) and substituting into (30), we obtain

$$(\pi^{H \to L})^* (\overline{\psi}^{H \to L})^* \omega_H = (\mathrm{p}_{1_{H \to L}})^* j^* \omega_{(H)}. \tag{32}$$

The map $j \circ \mathrm{p}_{1_{H \to L}}$ is related with ϕ by $j \circ \mathrm{p}_{1_{H \to L}} = p \circ \phi$, where p is the projection $p : T^*M|_{M_{(H)}} = T^*M_{(H)} \oplus N^*M_{(H)} \to T^*M_{(H)}$. That is, we have the following commutative diagram.

$$
\begin{array}{ccc}
s_{H \to L} & \overset{\phi}{\hookrightarrow} & T^*_{M_{(H)}} M \\
{\scriptstyle \mathrm{p}_{1_{H \to L}}} \downarrow & & \downarrow {\scriptstyle p} \\
\mathbf{J}_{(H)}^{-1}(0) & \overset{j}{\hookrightarrow} & T^*M_{(H)}
\end{array}
$$

Therefore Eq. (32) is equivalent to

$$(\pi^{H \to L})^* (\overline{\psi}^{H \to L})^* \omega_H = (\mathrm{p}_{1_{H \to L}})^* j^* \omega_{(H)} = \phi^* p^* \omega_{(H)}. \tag{33}$$

So in order to finish the proof of (29) it is sufficient to show that

$$p^* \omega_{(H)} = i_H^* \omega, \tag{34}$$

which will be done in local coordinates.

Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on M adapted to $M_{(H)}$, so that $\mathcal{U} \cap M_{(H)}$ is described by $x_{k+1} = \cdots = x_n = 0$. Let $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system on T^*M. Let Θ and $\Theta_{(H)}$ be the canonical one-forms respectively on T^*M and on $T^*M_{(H)}$. In these local coordinates the maps i_H and p are

$$i_H(x, \xi) = (x_1, \ldots, x_k, 0 \ldots, 0, \xi_1, \ldots, \xi_n)$$
$$p(x, \xi) = p(x_1, \ldots, x_k, 0 \ldots, 0, \xi_1, \ldots, \xi_n) = (x_1, \ldots, x_k, \xi_1, \ldots, \xi_k).$$

Then,

$$p^*\Theta_{(H)} = \sum_{i=1}^{k} \xi_i \, dx_i = \sum_{i=1}^{n} \xi_i \, dx_i \big|_{\text{span}\left\{ \frac{\partial}{\partial x_i}; \, 1 \le i \le k \right\}} = i_H^*\Theta,$$

and the result (34) follows for the respective symplectic forms by taking the exterior derivative. $\quad\square$

The previous proposition describes in part the abstract reduced symplectic form $\omega_0^{(L)}$ by means of natural explicitly constructed closed two-forms on each piece of the secondary decomposition. However, this is not a complete description since we cannot say what is $\omega_0^{(L)}$ at a point of a seam applied to vectors that are not tangent to that seam. The next theorem gives a characterization of the reduced form, as well as information on the symplectic data of the Σ-submanifolds that form the secondary decomposition.

Theorem 8. *In the conditions of Proposition 6, the reduced symplectic form $\omega_0^{(L)}$ of the symplectic piece $\mathcal{P}_0^{(L)}$ is the unique smooth extension of Ω_L from C_L to $\mathcal{P}_0^{(L)}$. Furthermore, the following are satisfied:*

1. *C_L is an open dense maximal symplectic Σ-submanifold of $(\mathcal{P}_0^{(L)}, \omega_0^{(L)})$ symplectomorphic to $(T^*M^{(L)}, \omega_L)$.*
2. *$S_{H \to L}$ are coisotropic Σ-submanifolds of $(\mathcal{P}_0^{(L)}, \omega_0^{(L)})$.*

Proof. Consider a point $x \in \mathcal{P}_0^{(L)}$ and two vectors $X_x, Y_x \in T_x \mathcal{P}_0^{(L)}$. Because C_L is open and dense we can find a sequence of points $x_k \in C_L$ and vectors $X_{x_k}, Y_{x_k} \in T_{x_k} C_L \simeq T_{x_k} \mathcal{P}_0^{(L)}$ such that

$$\lim x_k = x, \quad \lim X_{x_k} = X_x, \quad \lim Y_{x_k} = Y_x.$$

We can then study the existence of the limit of the sequence $\Omega_L(x_k)(X_{x_k}, Y_{x_k})$ as $k \to \infty$. By Proposition 6 we have that

$$\lim \Omega_L(x_k)(X_{x_k}, Y_{x_k}) = \lim \omega_0^{(L)}(x_k)(X_{x_k}, Y_{x_k}) = \omega_0^{(L)}(x)(X_x, Y_x),$$

where in the first equality we have used openness and density of C_L through the identification $T_{x_k} C_L \simeq T_{x_k} \mathcal{P}_0^{(L)}$, and the last equality comes from continuity of $\omega_0^{(L)}$. So, we have proved that there exists a unique continuous extension of Ω_L to $\mathcal{P}_0^{(L)}$. That this extension is smooth follows from the fact that $\omega_0^{(L)}$ is the extension and is known to be smooth by general reduction theory. The restrictions of this extension to C_L and to each seam follow tautologically from Proposition 6.

(1) is a trivial consequence of Theorem 7 and Proposition 6. To prove (2), first recall from symplectic linear algebra (see [11] for instance) that for (V, ω) a symplectic vector space and W a vector subspace, then W is coisotropic if and only if rank $(\omega|_W) = 2 \dim W - \dim V$.

In our case we will do this dimension counting with respect to the following tangent spaces. First fix $x \in S_{H \to L} \subset \mathcal{P}_0^{(L)}$ and let $y \in s_{H \to L}$ be such that $x = \pi^{H \to L}(y)$ and $G_y = L$. Note that we can always find such a y. Now denote by $z := t_{H \to L}(y)$ the projection of y to the base Σ-manifold $M_{(H)}$ so that $G_z \in (H)$. Let us call $H' := G_z$. Then we set $V = T_x \mathcal{P}_0^{(L)}$, $W = T_x S_{H \to L}$ and $\omega = \omega_0^{(L)}(x)$. Note that by Proposition 6 we have $\omega|_W = \Lambda_{H \to L}(x)$.

Now, since $T^*M^{(L)}$ is open and dense in $\mathcal{P}_0^{(L)}$,

$$\dim V = \dim T^*M^{(L)} = 2 (\dim M_{(L)} - \dim G + \dim L).$$

On the other hand, by construction of $\Lambda_{H \to L}$, we have that

$$\text{rank } \omega|_W = \dim T^*M^{(H)} = 2 (\dim M_{(H)} - \dim G + \dim H).$$

Finally, we have to compute $\dim W = \dim S_{H \to L}$. For this, note that $\dim S_{H \to L} = \dim (\mathbf{J}^{-1}(0) \cap T_z^* M)_{(L)} + \dim M_{(H)} - \dim G + \dim L$, where (L) refers to the linear H'-action. On the other hand, the Legendre transform maps $(\mathbf{J}^{-1}(0) \cap T_z^* M)_{(L)}$ H'-equivariantly isomorphically to $(S_z)_{(L)}$. Now, if ϕ and U are the diffeomorphism and neighborhood of z in M given by the Tube Theorem, then ϕ restricts to a diffeomorphism between $G \times_{H'} (S_z)_{(L)}$ and $U \cap M_{(L)}$. Since $\dim G \times_{H'} (S_z)_{(L)} = \dim G + \dim (S_z)_{(L)} - \dim H$, we can compute

$$\dim (S_z)_{(L)} = \dim M_{(L)} - \dim G + \dim H.$$

Finally, we obtain $\dim W = \dim M_{(H)} + \dim M_{(L)} - 2 \dim G + \dim H + \dim L$. It is then clear that the condition $\mathrm{rank}\,(\omega|_W) = 2 \dim W - \dim V$ is always satisfied. $\quad\square$

As a straightforward application of dimension counting we obtain the following result.

Corollary 9. *We have the following facts about seams:*

(i) *If $(H) \neq (L)$ the seam $S_{H \to L}$ can never be a symplectic submanifold of $\mathcal{P}_0^{(L)}$.*
(ii) *If $(H) \neq (L)$, the seam $S_{H \to L}$ is a coisotropic submanifold whose symplectic leaf space associated to the null foliation of $\Lambda_{H \to L}$ is symplectomorphic to $T^* M^{(H)}$ with its canonical symplectic form.*
(iii) *A connected component of $S_{H \to L}$ is a Lagrangian submanifold of $\mathcal{P}_0^{(L)}$ if and only if the corresponding connected component (i.e., under the projection $t^{H \to L}$) of $M^{(H)}$ is zero-dimensional.*

Proof. For (i), it is obvious that if $H \neq L$ then $\Lambda_{H \to L}$ has nonzero kernel. To see (ii), we know from the Theorem 8 that $S_{H \to L}$ is a coisotropic submanifold of $\mathcal{P}_0^{(L)}$ and that the restriction of the symplectic form to the seam satisfies

$$\omega_0^{(L)}|_{S_{H \to L}} = (\overline{\psi}^{H \to L})^* \omega_H,$$

where, recall, $\overline{\psi}^{H \to L} : S_{H \to L} \to T^* M^{(H)}$ is a surjective submersion. Since the symplectic leaf space is characterized by precisely this equation, (ii) follows. For (iii), note that $S_{H \to L}$ is coisotropic, so it is Lagrangian if and only if it has minimal dimension, i.e., $\frac{1}{2} \dim \mathcal{P}_0^{(L)} = \dim M^{(L)} = \dim M_{(L)} - \dim G + \dim L$. Recalling from the proof of the last theorem that $\dim S_{H \to L} = \dim M_{(H)} + \dim M_{(L)} - 2 \dim G + \dim H + \dim L$, we obtain that the Lagrangian condition is satisfied if $\dim M_{(H)} - \dim G + \dim H = 0$, but this is nothing but the dimension of $\dim M^{(H)}$. $\quad\square$

4.2. The coisotropic decomposition of $\mathcal{P}_0$

In this section we analyze the global structure of the topological space $\mathcal{P}_0$, describing a new, cotangent-bundle adapted, decomposition that is finer than the symplectic one. Recall from previous sections that for each isotropy class (L) in M there is a symplectic piece $\mathcal{P}_0^{(L)}$ in the reduced space, and the converse is also true. Furthermore, each of these pieces is again a Σ-decomposed space with an open and dense piece C_L, diffeomorphic to the cotangent bundle $T^* M^{(L)}$ and a collection of seams $S_{H \to L}$, one for each connectable pair $H \to L$ over (L) satisfying $(H) \neq (L)$. In this sense we obtained that the (L)-type symplectic piece of the zero momentum reduced space has the structure of a "topological fiber bundle" over $\overline{M^{(L)}}$, where the continuous projection τ^L is a Σ-decomposed surjective submersion.

We want now to extend this bundle picture to the whole reduced symplectic space $\mathcal{P}_0$. First of all, let $\tau_0 = \tau|_{\mathbf{J}^{-1}(0)}$ be the restriction of the cotangent bundle projection to the zero momentum level set, which is G-equivariant, and τ^0 the corresponding descended map $\tau^0 : \mathcal{P}_0 \to M/G$. By similar arguments to those in the previous section, τ^0 is a continuous surjective open map. It should be immediately noticed that it is not a morphism of Σ-decomposed spaces if $\mathcal{P}_0$ is endowed with the symplectic decomposition and M/G with the orbit type one, since by Theorem 7 the image of $\mathcal{P}_0^{(L)}$ is contained in the closure of $M^{(L)}$ and it has nonempty intersection with the boundary. It is our aim to explain how a different decomposition of $\mathcal{P}_0$ in terms of cotangent bundles and seams can be given in a way such that τ^0 is a Σ-decomposed surjective submersion. Consider the following partition of $\mathcal{P}_0$:

$$\mathcal{P}_0 = \bigsqcup_{(L)} C_L \bigsqcup_{(K') > (K)} S_{K' \to K} \quad \text{for all } (L), (K), (K') \in I_M. \tag{35}$$

Obviously τ^0 restricts on each piece to the previously defined smooth surjective submersions

$$\tau^0|_{C_L} = t^L : C_L \to M^{(L)} \quad \text{and} \quad \tau^0|_{S_{K' \to K}} = t^{K' \to K} : S_{K' \to K} \to M^{(K')}.$$

The next theorem explains the properties of this partition as well as the bundle structure of $\mathcal{P}_0$.

Theorem 10. *The partition* (35) *of* $\mathcal{P}_0$ *is a* Σ-*decomposition, that we will call coisotropic decomposition, and satisfies:*

1. *If* (H_0) *is the principal orbit type in* M *then* C_{H_0} *is open and dense in* $\mathcal{P}_0$.
2. *The frontier conditions are:*
    (i) $C_K \subset \partial C_H$ *if and only if* $(H) < (K)$.
    (ii) $S_{K \to H} \subset \partial C_H$ *if and only if* $(H) < (K)$.
    (iii) $C_K \subset \partial S_{K \to H}$ *if and only if* $(H) < (K)$.
    (iv) $S_{K' \to H} \subset \partial S_{K \to H}$ *if and only if* $(H) < (K) < (K')$.
    (v) $S_{K \to H'} \subset \partial S_{K \to H}$ *if and only if* $(H) < (H') < (K)$.
3. *The continuous projection* $\tau^0 : \mathcal{P}_0 \to M/G$ *is a* Σ-*decomposed surjective submersion with respect to the coisotropic decomposition of* $\mathcal{P}_0$ *and the usual orbit type decomposition of* M/G.
4. *If* I_M *has more than one class the coisotropic decomposition is strictly finer than the symplectic decomposition, otherwise they are identical.*

Proof. (1) Note that by Proposition 4 the symplectic Σ-decomposition of $\mathcal{P}_0$ has pieces $\mathcal{P}_0^{(L)}$ for every $(L) \in I_M$. So, if (H_0) is the principal orbit type in M then $\mathcal{P}_0^{(H_0)}$ is an open and dense Σ-submanifold of $\mathcal{P}_0$. As C_{H_0} is open and dense in $\mathcal{P}_0^{(H_0)}$ with respect to the topology in $\mathcal{P}_0^{(H_0)}$ and this topology is the induced one from the topology in $\mathcal{P}_0$, the result follows.

(2) The items (ii) and (iv) follow from Theorem 7 regarding the pieces in the statement as pieces of the decomposition of $\mathcal{P}_0^{(H)}$ for the respective (H). Also, (iii) follows from (v) by taking the limit $(H') = (K)$. Then, it remains to show (i) and (v).

For (i), recall that from the symplectic Σ-decomposition of $\mathcal{P}_0$ we have the following frontier conditions:

$$\mathcal{P}_0^{(K)} \subset \partial \mathcal{P}_0^{(H)} \iff (H) < (K).$$

As $C_K \subset \mathcal{P}_0^{(K)} \subset \partial \mathcal{P}_0^{(H)}$ if and only if $(H) < (K)$, then any open set V_x in $\mathcal{P}_0$ containing a point $x \in C_K$ must have nonempty intersection with $\mathcal{P}_0^{(H)}$ if and only if $(H) < (K)$. But, since C_H is dense in $\mathcal{P}_0^{(H)}$ it follows that V_x also intersects C_H, proving (i).

For (v), first note that a seam $S_{K \to H'}$ is only defined if $(H') < (K)$. Let $x \in S_{K \to H'} \subset \mathcal{P}_0^{(H')} \subset \mathcal{P}_0$ and U_x an open neighborhood of x in $\mathcal{P}_0$. So $\pi^{-1}(U_x)$ is an open neighborhood of a point $z \in \mathbf{J}^{-1}(0)$ such that $\pi(z) = x$, where π denotes the orbit projection, $\pi : \mathbf{J}^{-1}(0) \to P_0$.

As the point x projects under the map τ^0 to $m \in M^{(K)}$ then we can assume, without loss of generality, that $\tau(z) = y \in M_{(K)}$ satisfying $G_y = K$. From (13), the zero momentum level set restricted to the fiber over y is given by

$$\mathbf{J}_y^{-1}(0) = \mathbf{J}_{(K)y}^{-1}(0) \oplus N_y^* M_{(K)}.$$

Now note that because $\pi(z) = x \in S_{K \to H'}$ then

$$z \in \mathbf{J}_{(K)y}^{-1}(0) \times (N_y^* M_{(K)})_{(H')}$$

where the orbit type on the conormal fiber refers to the linear K action. Recall from the orbit type decomposition of the conormal fiber $N_y^* M_{(K)}$ that for any $(H) \in I_M$ such that $(H) < (K)$ then $(N_y^* M_{(K)})_{(H)} \neq \emptyset$, and consequently $(N_y^* M_{(K)})_{(H')} \subset \partial (N_y^* M_{(K)})_{(H)}$ if $(H) < (H') < (K)$. This means that there is a point $z' \in \pi^{-1}(U_x) \cap (\mathbf{J}_{(K)y}^{-1}(0) \times (N_y^* M_{(K)})_{(H)})$, from where (v) easily follows once we note that $\pi(z') \in U_x \cap S_{K \to H}$.

(3) follows from the definition of a Σ-decomposed surjective submersion, since $\tau^0|_{C_L} = t^L$ and $\tau^0|_{S_{K' \to K}} = t^{K' \to K}$ are surjective submersions and the pieces of the coisotropic decomposition of $\mathcal{P}_0$ are the C_L's and the seams, and the pieces of the orbit type decomposition of M/G are $M^{(L)}$ for every $(L) \in I_M$.

Finally, (4) is obvious from the construction of the coisotropic and symplectic decompositions. $\square$

From the frontier conditions (i) to (iii) it is clear that two cotangent bundles C_K and C_H are stitched along the corresponding seam $S_{K \to H}$. The pieces of the coisotropic decomposition are in one-to-one correspondence with the connectable pairs of I_M, where to a connectable pair of two copies of a same class $H \to H$ corresponds the cotangent bundle C_H, and for different classes $K \to H$, $(H) \neq (K)$ the corresponding piece is a seam $S_{K \to H}$. Thus Theorem 10 allows us to obtain the coisotropic decomposition lattice with only the knowledge of the lattice I_M.

5. From Σ-decompositions to stratifications

It was the objective of this paper to give a description of the topology and geometry of the reduced space $\mathcal{P}_0$, and for a number of important reasons such a description based in the stratified nature of the singular spaces involved is more desirable than the one based only in the weaker concept of Σ-decompositions. In this section we upgrade our previous topological results and in the following we will concentrate on giving meaning and justification to the following assertion:

Theorem 11. *All the Σ-decomposed spaces in Theorems 7 and 10 are stratified spaces with the unique stratifications induced by their Σ-decompositions. Consequently all the maps involved are morphisms of stratified spaces. In particular τ^0 and τ^L are stratified surjective submersions.*

We need then an appropriate definition of stratification and morphism of stratified spaces. We will follow closely Ref. [21] for the definitions in the rest of the section. We caution the reader that other authors use different definitions for the same terminology (for example the definition of stratification found in [24], which also includes the extra properties of being a cone space). Let X be a topological space and $\mathcal{S}$ a map that associates to each point $x \in X$ the set germ $\mathcal{S}_x$ at x of a locally closed subset of X. Recall that the set germ of a set A at $x \in A$ is the equivalence class $[A]_x$ of A at x defined by $[A]_x = [B]_x$ if both A and B are subsets of X containing x and such that there exists an open neighborhood U of x satisfying $A \cap U = B \cap U$.

From now on we shall call a Σ-decomposition for which, given any piece, all its connected components have the same dimension, a *decomposition*.

Definition 3. In the previous conditions, the map $\mathcal{S}$ is said to be a stratification of X if, for any point $x \in X$, there exists an open neighborhood U containing x and a decomposition $\mathcal{Z}$ of U satisfying: For any $y \in U$, $\mathcal{S}_y = [Z]_y$, with $Z \in \mathcal{Z}$ the piece containing y. The pair $(X, \mathcal{S})$ is called a stratified space.

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be two stratified spaces and $f : X \to Y$ a continuous map between the underlying topological spaces. f is called a morphism of stratified spaces if for every $x \in X$ there exist neighborhoods V of $f(x)$ and $U \subset f^{-1}(V)$ of x with decompositions $\mathcal{X}$ and $\mathcal{Y}$ inducing $\mathcal{S}|_U$ and $\mathcal{T}|_V$ respectively, such that for every $x' \in U$ there is an open neighborhood $W \subset U$ containing x' such that the restriction $f|_W$ maps the intersection of the piece S containing x' with W into a piece $R \in \mathcal{Y}$ and $f|_{S \cap W} : S \to R$ is smooth.

We will say that f is a stratified immersion (resp. submersion, diffeomorphism, etc.) if so are all the restrictions $f|_{S \cap W}$ at every point $x \in X$.

Obviously if $(X, \mathcal{X})$ is a decomposed space, for any neighborhood U of any point, $(U, \mathcal{X}|_U)$ is again a decomposed space, and then we can give X the structure of a stratified space associating to each of its points x the set germ of the piece containing x. This stratification is said to be induced by the decomposition $\mathcal{X}$. As an immediate consequence a morphism of decomposed spaces is a morphism of the induced stratified spaces.

A Σ-decomposition $\mathcal{X}$ in principle does not induce a stratification, since $\mathcal{X}|_U$ could be a Σ-decomposition instead of a decomposition of U no matter how U is chosen, as we can see in the following example: Consider the subspace X of $\mathbb{R}^3$ given by the (x_1, x_2)-plane and the x_3-axis. Let $X_1 = X \setminus \mathbf{0}$, $X_2 = \mathbf{0}$. Obviously X_1 and X_2 are Σ-manifolds and the partition $X = X_1 \cup X_2$ is a Σ-decomposition of X, but for any open neighborhood U of $\mathbf{0}$ the induced partition of U is again a Σ-decomposition, so the map associating to each point the equivalence class of the piece containing it is not a stratification.

However, in the special case of the orbit type Σ-decomposition of a proper G-manifold M it is possible to induce a decomposition of a suitable open neighborhood of an arbitrary point. Furthermore, it is possible to guarantee that the secondary and coisotropic Σ-decompositions induced from the orbit type one are locally decompositions, fulfilling the requirements for inducing stratifications, for which the decomposed morphisms are automatically stratified morphisms.

The reason for this lies once again in the local model of an invariant neighborhood U of an orbit $G \cdot m$ given by the tubular neighborhood $G \times_H S_m$, where $G_m = H$ and S_m is a linear slice orthogonal to the directions tangent to the orbit at m. In this model the orbit type $U_{(L)}$ is represented by $G \times_H (S_m)_{(L)}$, where L must be a subgroup of H and the action on the linear slice is the linear H-action by isometries with respect to the restriction of the inner product in $T_m M$. But it is known that the partition of a vector space by orbit types with respect to the linear representation of a compact Lie group is a decomposition (see for instance Lemma 4.10.12 of [4]). Consequently the induced Σ-decomposition of U, consisting of the intersection of pieces in M with U, is actually a decomposition since the pieces are of the form $G \times_H (S_m)_{(L)}$, its connected components having the same dimension.

To see that the coisotropic decomposition is a stratification, first recall that the map $\tau^0 : \mathcal{P}_0 \to M/G$ is an open, Σ-decomposed map. Now, choosing a suitable small enough open set, U in $\mathcal{P}_0$, it will project to a decomposed open set, $O := \tau^0(U)$, where all pieces have components of the same dimension, since M/G is locally decomposed. A connected component of $S_{H \to L} \cap U$ projects under τ^0 to a connected component of $M^{(H)} \cap O$, and its dimension is determined by the dimension of this component of $M^{(H)} \cap O$ and the dimension of some other connected component of $M^{(L)} \cap O$ as we have seen in the proof of Theorem 8. Since O is a decomposed space then all these pieces of the form $M^{(L)} \cap O$ have the same dimension, from where it follows that all the connected components of $S_{H \to L} \cap U$ have the same dimension, and therefore U is a decomposed open set in $\mathcal{P}_0$, proving that the coisotropic decomposition is a stratification. Similar arguments work for the secondary decomposition, and so we conclude Theorem 11. We are therefore justified to use the terminology secondary and coisotropic stratifications, as well as their corresponding stratification lattices.

6. An example

We will illustrate the main results obtained in this paper with an example that is simple, yet rich enough to show the extra structure appearing in singular symplectic reduction for cotangent bundles. We will compute the secondary and coisotropic stratifications exhibiting explicitly the corresponding frontier conditions predicted in Theorems 7 and 10.

Consider the $G = \mathbb{Z}_2 \times S^1$ action on $M = \mathbb{R}^3$, where S^1 acts by rotations around the x_3-axis and $\mathbb{Z}_2$ by reflections with respect to the plane (x_1, x_2). The isotropy lattice and the decomposition lattice for this action are shown in Fig. 1. Let $\mathbb{R}^3$ be equipped with the Euclidean inner product which defines a G-invariant Riemannian metric for this action. Identifying $T^*\mathbb{R}^3$ with $\mathbb{R}^3 \times \mathbb{R}^3$ then the cotangent lifted action is diagonal, $g \cdot (v_1, v_2) = (g \cdot v_1, g \cdot v_2)$ for $g \in G$ and $v_1, v_2 \in \mathbb{R}^3$. Let $(x_1, x_2, x_3, y_1, y_2, y_3)$ be the coordinates of the vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3$ with respect to the canonical basis.

The ring of G-invariant polynomials, $\mathcal{P}^G(\mathbb{R}^3 \times \mathbb{R}^3)$, is generated by

$$\begin{aligned}
&\sigma_1 = x_1^2 + x_2^2 + y_1^2 + y_2^2, \quad \rho_1 = x_3^2 + y_3^2, \\
&\sigma_2 = 2(x_1 y_1 + x_2 y_2), \quad \rho_2 = 2 x_3 y_3, \\
&\sigma_3 = y_1^2 + y_2^2 - x_1^2 - x_2^2, \quad \rho_3 = y_3^2 - x_3^2, \\
&j = x_1 y_2 - x_2 y_1.
\end{aligned}$$

These polynomials are subject to the relations

$$\sigma_1 \geq 0, \quad \rho_1 \geq 0, \quad \sigma_1^2 = \sigma_2^2 + \sigma_3^2 + 4j^2, \quad \rho_1^2 = \rho_2^2 + \rho_3^2.$$

Note that the relations between the σ's and the ρ's are uncoupled if j is zero. The momentum map for the cotangent lifted action of G is $\mathbf{J}(\mathbf{x}, \mathbf{y}) = j$.

Let now Z be a G-invariant subset of $\mathbb{R}^3 \times \mathbb{R}^3$ such that j is constant on Z. Consider two copies of $\mathbb{R}^3$, which will be denoted by $\mathbb{R}^{3\sigma}$ and $\mathbb{R}^{3\rho}$ and the maps $\chi_\sigma : Z \to \mathbb{R}^{3\sigma}$ and $\chi_\rho : Z \to \mathbb{R}^{3\rho}$ defined as

$$\chi_\sigma(z) = (\sigma_1(z), \sigma_2(z), \sigma_3(z)), \quad \chi_\rho(z) = (\rho_1(z), \rho_2(z), \rho_3(z))$$

for every $z \in Z$. The Hilbert map $\chi := (\chi_\sigma, \chi_\rho) : Z \to \mathbb{R}^{3\sigma} \times \mathbb{R}^{3\rho}$ is G-invariant, and due to the relation between the polynomials its image Im χ = Im χ_σ × Im χ_ρ $\in \mathbb{R}^{3\sigma} \times \mathbb{R}^{3\rho}$ is a topological space equipped with the relative topology which is a semi-algebraic variety. The Tarski–Seidenberg Theorem (see [7] and references therein for a more detailed explanation) gives that Im χ has a canonical (Whitney) stratification. By invariant theory the map χ restricts
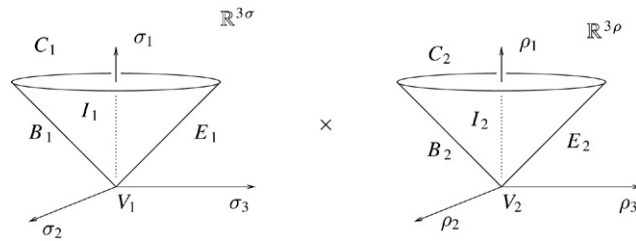
Fig. 2. The reduced singular space $\mathcal{P}_0$ as a product of two cones.

to a homeomorphism $\overline{\chi} : Z/G \to \operatorname{Im} \chi_\sigma \times \operatorname{Im} \chi_\rho \in \mathbb{R}^{3\sigma} \times \mathbb{R}^{3\rho}$ that happens to be an isomorphism of stratified spaces if Z/G is endowed with the orbit type stratification. In order to apply the results obtained in previous sections we will study the case $Z = \mathbf{J}^{-1}(0)$ through the image of χ.

The zero level set of the momentum map is $Z = \mathbf{J}^{-1}(0) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 \mid j(\mathbf{x}, \mathbf{y}) = 0\}$. So we can identify $\mathcal{P}_0$ with the direct product of the two cones defined by the relations

$$C_1 : \sigma_1^2 = \sigma_2^2 + \sigma_3^2, \quad \text{and} \quad C_2 : \rho_1^2 = \rho_2^2 + \rho_3^2.$$

This realization of $\mathcal{P}_0$ is shown in Fig. 2. For future reference, in Fig. 2 we mark some subsets on each of the cones. For instance in C_1 the vertex is marked as V_1, the straight line $\sigma_1 = \sigma_3$ excluding the origin is labelled E_1, the opposite line $\sigma_1 = -\sigma_3$ also, except the origin is labelled as B_1, and finally all the cone except $V_1 \cup E_1$ is called I_1 (I_1 contains B_1). Note from the defining equation of C_1 that B_1 and E_1 form an angle of $\pi/2$. Analogous definitions apply to C_2.

By Proposition 4 we know that the orbit types present in Z are exactly those which are present in M, i.e., the elements of I_M. This implies that the symplectic strata of $\mathcal{P}_0$ are in one-to-one correspondence with the strata of M, and that both spaces exhibit an identical stratification lattice as we will verify now. Indeed, studying the diagonal action restricted to Z one finds easily the following orbit types:

$$Z_{(\mathbb{Z}_2 \times S^1)} = \{(\mathbf{0}, \mathbf{0})\}$$
$$Z_{(\mathbb{Z}_2)} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 | x_3 = y_3 = 0, x_1 y_2 - x_2 y_1 = 0)\},$$
$$Z_{(S^1)} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 \mid x_1 = x_2 = y_1 = y_2 = 0, (x_3, y_3) \neq (0, 0)\}$$
$$Z_{(1)} = Z \setminus (Z_{(\mathbb{Z}_2 \times S^1)} \cup Z_{(\mathbb{Z}_2)} \cup Z_{(S^1)}).$$

Using the image of the map χ we have

$$\mathcal{P}_0^{(\mathbb{Z}_2 \times S^1)} = V_1 \times V_2$$
$$\mathcal{P}_0^{(\mathbb{Z}_2)} = (I_1 \cup E_1) \times V_2 = (C_1 \setminus V_1) \times V_2$$
$$\mathcal{P}_0^{(S^1)} = V_1 \times (I_2 \cup E_2) = V_1 \times (C_2 \setminus V_2)$$
$$\mathcal{P}_0^{(1)} = (I_1 \cup E_1) \times (I_2 \cup E_2) = (C_1 \setminus V_1) \times (C_2 \setminus V_2).$$

The above sets are the strata of the symplectic stratification lattice predicted by Theorem 2. This lattice is shown in Fig. 3(a).

Recall that the strata for the secondary stratification of each symplectic stratum $\mathcal{P}_0^{(L)}$ are of two types, cotangent bundles C_L and seams $S_{H \to L}$ with $(H) > (L)$ defined by (17). Let us now study the secondary stratification of each symplectic stratum in $\mathcal{P}_0$. We embed $M = \mathbb{R}^3$ in T^*M by the injection

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0, 0, 0). \tag{36}$$

We then have

$$T^*M_{(\mathbb{Z}_2 \times S^1)} = \{(\mathbf{0}, \mathbf{0})\}$$
$$T^*M_{(\mathbb{Z}_2)} = \{(x_1, x_2, 0, y_1, y_2, 0), (x_1, x_2) \neq (0, 0)\}$$
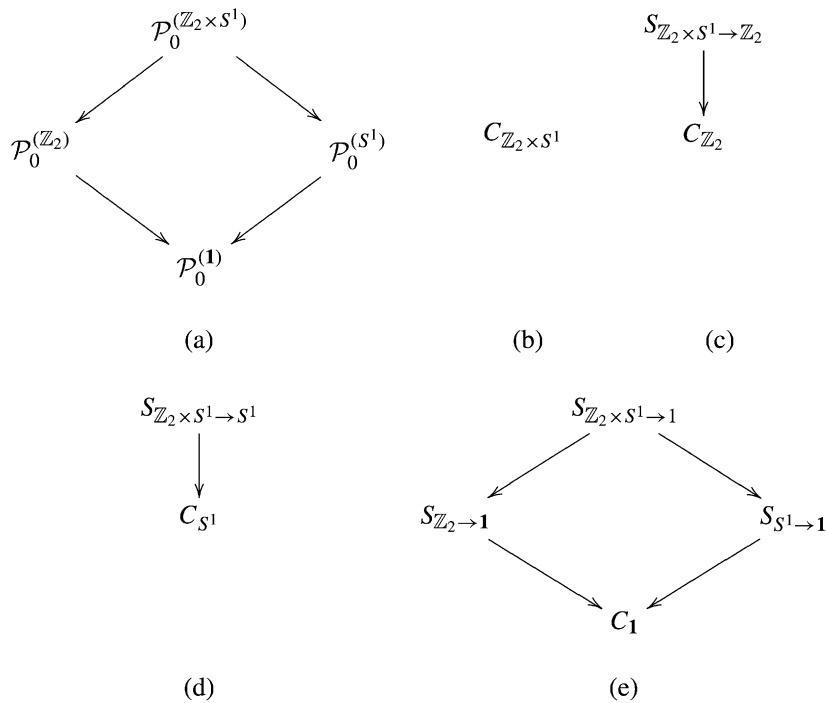$$T^*M_{(S^1)} = \{(0, 0, x_3, 0, 0, y_3), x_3 \neq 0\}$$

Fig. 3. (a) Symplectic stratification of $\mathcal{P}_0$. Secondary stratifications of: (b) $\mathcal{P}_0^{(\mathbb{Z}_2 \times S^1)}$, (c) $\mathcal{P}_0^{(\mathbb{Z}_2)}$, (d) $\mathcal{P}_0^{S^1}$ and (e) $\mathcal{P}_0^{(\mathbf{1})}$.

$$T^*M_{(\mathbf{1})} = \{(\mathbf{x}, \mathbf{y}), \mathbf{x} \in M_{(\mathbf{1})}\}$$
$$N^*M_{(\mathbb{Z}_2 \times S^1)} = \{(\mathbf{0}, \mathbf{y}), \mathbf{y} \in \mathbb{R}^3\}$$
$$N^*M_{(\mathbb{Z}_2)} = \{(x_1, x_2, 0, 0, 0, y_3), (x_1, x_2) \neq (0, 0)\}$$
$$N^*M_{(S^1)} = \{(0, 0, x_3, y_1, y_2, 0), x_3 \neq 0\}$$
$$N^*M_{(\mathbf{1})} = \{(\mathbf{x}, \mathbf{0}), \mathbf{x} \in M_{(\mathbf{1})}\}.$$

Computing the seams and the cotangent bundles we obtain the following realization of these two types of pieces in the image of χ:

$$
\begin{array}{ll}
C_{\mathbb{Z}_2 \times S^1} = V_1 \times V_2 & S_{\mathbb{Z}_2 \times S^1 \to \mathbb{Z}_2} = E_1 \times V_2 \\
C_{\mathbb{Z}_2} = I_1 \times V_2 & S_{\mathbb{Z}_2 \times S^1 \to S^1} = V_1 \times E_2 \\
C_{S^1} = V_1 \times I_2 & S_{\mathbb{Z}_2 \times S^1 \to \mathbf{1}} = E_1 \times E_2 \\
C_{\mathbf{1}} = I_1 \times I_2 & S_{\mathbb{Z}_2 \to \mathbf{1}} = I_1 \times E_2 \\
& S_{S^1 \to \mathbf{1}} = E_1 \times I_2.
\end{array}
$$

According to the results of Theorem 7 the secondary stratifications of the symplectic strata are:

$$\mathcal{P}_0^{(\mathbb{Z}_2 \times S^1)} = C_{(\mathbb{Z}_2 \times S^1)} = V_1 \times V_2$$
$$\mathcal{P}_0^{(\mathbb{Z}_2)} = C_{\mathbb{Z}_2} \cup S_{\mathbb{Z}_2 \times S^1 \to \mathbb{Z}_2} = (I_1 \times V_2) \cup (E_1 \times V_2)$$
$$\mathcal{P}_0^{(S^1)} = C_{S^1} \cup S_{\mathbb{Z}_2 \times S^1 \to S^1} = (V_1 \times I_2) \cup (V_1 \times E_2)$$
$$\mathcal{P}_0^{(\mathbf{1})} = C_{\mathbf{1}} \cup S_{\mathbb{Z}_2 \to \mathbf{1}} \cup S_{S^1 \to \mathbf{1}} \cup S_{\mathbb{Z}_2 \times S^1 \to \mathbf{1}} = (I_1 \times I_2) \cup (I_1 \times E_2) \cup (E_1 \times I_2) \cup (E_1 \times E_2).$$

The corresponding stratification lattices are shown in Fig. 3(b)–(e). The coisotropic stratification lattice is shown in Fig. 4. These lattices are constructed using the results of Theorems 7 and 10, and the corresponding frontier conditions can be verified from the above expressions. We describe now the bundle structure of these stratifications: using Eq. (36) we realize the quotient M/G as the subset of the image of χ given by $(B_1 \cup V_1) \times (B_2 \cup V_2)$. The corresponding
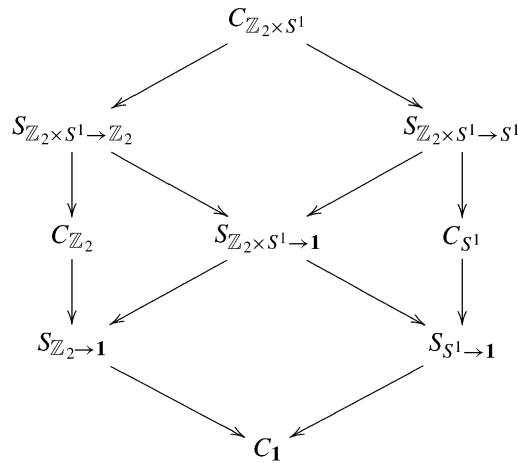
$$C_{\mathbb{Z}_2 \times S^1}$$

$$S_{\mathbb{Z}_2 \times S^1 \to \mathbb{Z}_2} \qquad S_{\mathbb{Z}_2 \times S^1 \to S^1}$$

$$C_{\mathbb{Z}_2} \qquad S_{\mathbb{Z}_2 \times S^1 \to \mathbf{1}} \qquad C_{S^1}$$

$$S_{\mathbb{Z}_2 \to \mathbf{1}} \qquad S_{S^1 \to \mathbf{1}}$$

$$C_{\mathbf{1}}$$

Fig. 4. Coisotropic stratification of $\mathcal{P}_0$.

strata of its orbit type stratification are:

$$M^{(\mathbb{Z}_2 \times S^1)} = V_1 \times V_2 \quad M^{(\mathbb{Z}_2)} = B_1 \times V_2$$
$$M^{(S^1)} = V_1 \times B_2 \qquad M^{(\mathbf{1})} = B_1 \times B_2.$$

The map $\tau^0 : \mathcal{P}_0 \to M/G$ is obtained as follows. Let $z = (\mathbf{x}_1, \mathbf{x}_2) \in C_1 \times C_2$ be a point of $\mathcal{P}_0$, then $\tau^0(z)$ is a point $(\mathbf{b}_1, \mathbf{b}_2) \in B_1 \times B_2$, where $\mathbf{b}_1$ is the point in the intersection of B_1 and the unique parabola obtained by sectioning the cone C_1 with a plane orthogonal to B_1 at $\mathbf{x}_1$. Analogously one defines in this way the point $\mathbf{b}_2 \in C_2$.

7. Final remarks

We have studied the global picture of two new stratifications of the zero momentum singular reduced space for a cotangent lifted action. The results obtained raise several natural questions which have not been addressed in this work.

First, as mentioned in the Introduction, it would be interesting to determine if these reduced spaces, together with the secondary and coisotropic stratifications, have conical structure, satisfy Whitney conditions and/or admit singular atlases and smooth structures, as it happens for the symplectic stratification (see [24] and [18]). For this, the Symplectic Slice Theorem of Marle, Guillemin and Sternberg is too weak, and a cotangent bundle adapted version of it, which could detect the secondary and coisotropic strata, would be needed. Unfortunately, such a technology does not yet exist in full generality. Some steps have been done in [23] but it is still lacking a general result. We expect however that advances in this line of research can lead to the proof that the secondary and the coisotropic stratifications enjoy the conical property and satisfy Whitney conditions like the symplectic stratification.

A different direction of study consists of describing reduction at nonzero momentum. At least for reduction at momentum values with trivial coadjoint orbits it is also possible to obtain a secondary and coisotropic stratification with some modifications of the technology used here. This will appear elsewhere. For general momenta the problem is much more involved, since the coadjoint representation interacts with the action on the base manifold to produce an isotropy lattice of the momentum level set $\mathbf{J}^{-1}(\mu)$. These are aspects of ongoing work on the subject.

Even when the secondary stratification of each symplectic stratum is not invariant for the reduced Hamiltonian flow, the fact that it captures the bundle structure of the reduced space might be useful for understanding certain qualitative aspects of the reduced dynamics. A typical situation would be described by the following observation: Even when it is known that the isotropy is preserved by the reduced dynamics (and hence the symplectic strata are dynamically invariant), this is not true for the isotropy of the projected dynamics onto M/G. It is precisely when the Hamiltonian evolution crosses different seams within its ambient symplectic stratum that changes in the isotropies of the base points occur. Natural questions arise then, like under what conditions symmetric Hamiltonian flows preserve isotropy both in phase and configuration spaces, or when a Hamiltonian evolution is maximal, in the sense that every

secondary stratum (and hence every possible isotropy type in the base) is crossed. This will be the object of further research.

Acknowledgements

References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, Second edition, Addison-Wesley Pub. Comp. Inc, 1987.

[2] J. Arms, R.H. Cushman, M.J. Gotay, A universal reduction procedure for Hamiltonian group actions, in: The geometry of Hamiltonian Systems, Berkeley, CA, 1989, in: Math. Sci. Res. Inst. Publ., vol. 22, Springer, New York, 1991, pp. 33–51.

[3] L. Bates, E. Lerman, Proper group actions and symplectic stratified spaces, Pacific J. Math. 181 (2) (1997) 201–229.

[4] P. Chossat, R. Lauterbach, Methods in Equivariant Bifurcations and Dynamical Systems, in: Advanced Series in Nonlinear Dynamics, vol. 15, World Scientific Publishing Co. Inc, River Edge, NJ, 2000.

[5] R.H. Cushman, L. Bates, Global Aspects of Classical Integrable Systems, Birkhäuser Verlag, Basel, 1997.

[6] R.H. Cushman, J. Śniatycki, Differential structure of orbit spaces, Canad. J. Math. 53 (4) (2001) 715–755.

[7] J.J. Duistermaat, J.A.C. Kolk, Lie Groups, in: Universitext, Springer-Verlag, 2000.

[8] C. Emmrich, H. Römer, Orbifolds as configuration spaces of systems with gauge symmetries, Commun. Math. Phys. 129 (1990) 69–94.

[9] J.L. Koszul, Sur certains groupes de transformation de Lie, Colloque de Gèometrie Différentielle, Colloques du CNRS 71 (1953) 137–141.

[10] E. Lerman, R. Montgomery, R. Sjamaar, Examples of Singular Reduction, in Symplectic Geometry, in: London Math. Soc. Lecture Note Ser., vol. 192, Cambridge Univ. Press, 1993, pp. 127–155.

[11] P. Libermann, C.M. Marle, Symplectic Geometry and Analytical Mechanics, in: Mathematics and its Applications, vol. 35, D. Reidel Publishing Co, Dordrecht, 1987.

[12] J.E. Marsden, Lectures on Mechanics, in: Lecture Note Series, vol. 174, LMS, Cambridge University Press, 1992.

[13] J.E. Marsden, M. Perlmutter, The orbit bundle picture of cotangent bundle reduction, C. R. Math. Acad. Sci. Soc. R. Can. 22 (2) (2000) 35–54.

[14] J.E. Marsden, T.S. Ratiu, Reduction of Poisson manifolds, Lett. Math. Phys. 11 (2) (1986) 161–169.

[15] J.E. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1) (1974) 121–130.

[16] K. Meyer, Symmetries and integrals in mechanics, in: Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, 1973, pp. 259–272.

[17] R. Montgomery, The structure of reduced cotangent phase spaces for non-free group actions, preprint 143 of the U.C. Berkeley Center for Pure and App. Math., 1983.

[18] J.-P. Ortega, T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, in: Progress in Mathematics, vol. 222, Birkhauser-Verlag, 2003.

[19] J.-P. Ortega, T.S. Ratiu, Singular reduction of Poisson manifolds, Lett. Math. Phys. 46 (1998) 359–372.

[20] R.S. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. 73 (1961) 295–323.

[21] M.J. Pflaum, Analytic and Geometric Study of Stratified Spaces, in: Lecture Notes in Mathematics, vol. 1768, Springer-Verlag, 2001.

[22] W.J. Satzer, Canonical reduction of mechanical systems invariant under abelian group actions with an application to celestial mechanics, Indiana Univ. Math. J. 26 (1977) 951–976.

[23] T. Schmah, Symmetries of cotangent bundles, Ph.D. Thesis, vol. 2443, École Polytechnique Fédérale de Lausanne (Suisse), 2002.

[24] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math. 134 (1991) 375–422.